行政院國家科學委員會專題研究計劃成果報告

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* 數值方法與壹週期函數的研究 *

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計劃類別：☑ 個別計劃 ☐ 整合型計劃

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執行單位：淡江大學數學系
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中文摘要:

在此報告中我們指出若一微分同胚函數與一自治式微分方程式之週期壹函數夠接近的話，則其可視為一非自治式微分方程式之週期壹函數. 我們並指出當我們用數值方法來估計自治式微分方程式之解時，其所得結果往往為非自治式微分方程式之解之現象.

關鍵詞：

微分同胚，週期函數，週期式微分方程，數值方法.

英文摘要:

It is shown that diffeomorphism are equivalent to time-one maps of one-time periodic differential equations. As an application, we show how this is related to numerical schemes.

關鍵詞：

diffeomorphism, time-one map, periodic differential equation, numerical scheme.

1 Main Result

Consider a system of autonomous ordinary differential equations in a domain $D \subset R^N$:

$$\dot{x} = f(x) \tag{1.1}$$

where $x \in D \subset R^N$, $f : D \to D$ is C^r, for some $r \geq 1$ and $f(0) = 0$.

Let $\varphi(t, a)$ denote the flow of (1.1) with $\varphi(0, a) = a$. We assume that D is bounded and the flow:

$$\varphi : R \times D \to D$$

is well defined and C^r. Let

$$\mathcal{B}C^r = \{f \mid f : D \to D \ \text{ is } \ C^r \text{with } \ \|f\|_r < \infty\}$$

here $\|\cdot\|_r$ is the usual C^r norm . The subset of all C^r diffeomorphisms in $\mathcal{B}C^r$ will be denoted by $\mathcal{D}^r$. Observe that the time one map of the flow $\varphi(1, \cdot)$ is in $\mathcal{D}^r$. Before stating the main result , we would like to point out that if the flow is being considered on a compact manifold, then the following theorem is still true.

Theorem 1.1

Suppose that $f \in \mathcal{B}C^r$ and $F \in \mathcal{D}^r$, where $r \geq 1$. Then there exists a constant $\delta > 0$ such that if $\|F(\cdot) - \varphi(1, \cdot)\|_1 < \delta$, then there exists a C^r non–autonomous time periodic vector field g, $g(t, x) = g(t + 1, x)$, such that the flow $\psi(t, a)$ defined by

$$\dot{x} = g(t, x) \tag{1.2}$$

satisfies $\psi(0, a) = a$ and $\psi(1, a) = F(a)$. In other words, F(a) is realized as the time-one map of the flow of equation (1.2).

2 Numerical Schemes

In this section, we will consider the pth order discretization with step size $h > 0$ of equation (1.1), where $p \geq 1$ and their realizations as maps defined by flows. In general, such discretization have the following formats:

$$x_{n+1} = H(x_n) = x_n + h\Phi(h, x_n), \quad n = 0, 1, 2, \cdots. \tag{2.1}$$

where $\Phi(h, x)$ is assumed to be C^r for some $r \geq 1$. For example, if $\Phi(h, x) = f(x)$, then the scheme given by H in (2.1) is the Euler's method. In this case, if $\varphi(t, a)$ is the flow of (1.1) satisfying $\varphi(0, a) = a$ then one has

$$\varphi(h, a) = a + hf(a) + O(|h|^2),$$

this shows that

$$|\varphi(h, a) - H(a)| = O(|h|^2).$$

On the other hand, it can be easily seen that

$$|\frac{\partial \varphi}{\partial a}(h, a) - H'(a)| = O(|h|).$$

Now consider the time rescaling variable $t = h\tau$ and denoting $x' = \frac{dx}{d\tau}$,then equation (1.1) can be transformed into the following form

$$x' = hf(x)$$

with flow $\pi(\tau, a)$. It is clear that $H(a) = \pi(1, a) + O(|h|^2)$. Thus, if $|h| \ll 0$, then as a consequence of theorem 2.1 , $H(a)$ can be realized as the time-1 map of a 1-periodic (in time τ) differential equation:

$$x' = hg(\tau, x).$$

In terms of the original time variable t, $H(x)$ is realized as the time-h map of the flow defined by the following equation:

$$\dot{x} = g(\frac{t}{h}, x)$$

For other numerical scheme, it is a consequence of the definition of pth order approximation that $H(a)$ satisfies the following condition for all $x \in D$:

$$|\varphi(h,a) - H(a)| \leq C|h|^p$$

where $C > 0$ is a constant. Thus, H can be made as close as possible to the flow itself by taking a small enough step size $h > 0$. However as stated in theorem 2.1 that we need C^1 closeness for the theorem to be valid. Thus , we need the following lemma.

Lemma 2.1
Suppose that $f(\cdot)$, $H(\cdot) \in \mathcal{B}C^r$, where $r \geq 2$. Then there exists h_0, $1 > h_0 > 0$ and $C_1 > 0$ such that

$$|\frac{\partial\varphi(h,a)}{\partial a} - H'(x)| \leq C_1|h| \tag{2.2}$$

for all $x \in D$ and h, $h_0 > h > 0$.

Theorem 2.2
Suppose that $f(\cdot), H(\cdot) \in \mathcal{B}C^r$, where $r \geq 2$. Then there exit h_0, $1 > h_0 > 0$ and a 1-periodic differential equation:

$$\dot{x} = g(\frac{t}{h}, x)$$

such that $f(x) \equiv g(0,x) \equiv g(1,x)$ and H can be realized as the time-h map of the above periodic system.

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