Positive Solutions of Systems of Second Order Differential Equations on a Measure Chain

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Abstract

The existence of nonnegative solutions of boundary value problems of the form $u^{\Delta\Delta}(t) + \lambda a(t)f(u^{\sigma}(t), v^{\sigma}(t)) = 0, v^{\Delta\Delta}(t) + \lambda b(t)g(u^{\sigma}(t), v^{\sigma}(t)) = 0, t \in [0,1]$, satisfying the boundary condition $u(0) = 0 = u^{\sigma}(1), v(0) = 0 = v^{\sigma}(1)$, where all functions are assumed to be continuous and nonnegative. It is shown that under reasonable growth conditions imposed on f and g there exists a nonnegative solution (u, v) for every $\lambda > 0$.

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1 Introduction

The theory of measure chains has recently received a lot of attention, since Hilger's [10] initial paper introduced the unifying theory for continuous and discrete calculus. Subsequent developmental major works devoted to the calculus on measure chains have been conducted by Agarwal and Bohner [1], Aublback and Hilger [2], Erbe and Hilger [3], Erbe and Peterson [4], and Kaymakcalan *et al.* [11]. Before introducing the problems of interest for this paper, we present some definitions and notation which are common to the recent literature. Our sources for this background material are the two papers by Erbe and Peterson [4], [5].

Definitions 1.1 *Let T be a closed subset of R, and let T have the subspace topology inherited from the Euclidean topology on R. For $t < \sup T$ and $r > \inf T$, define the* forward jump operator*, σ, and the* backward jump operator*, ρ, respectively, by*

$$\sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T,$$

$$\rho(r) = \sup\{\tau \in T \mid \tau < r\} \in T,$$

for all $t, r \in T$. If $\sigma(t) > t$, t is said to be right scattered*, and if $\rho(r) < r$, r is said to be* left scattered*. If $\sigma(t) = t$, t is said to be* right dense*, and if $\rho(r) = r$, r is said to be* left dense*.*

Definitions 1.2 *For $x : T \to R$ and $t \in T$ (if $t = \sup T$, assume t is not left scattered), define the* delta derivative *of $x(t)$, $x^{\Delta}(t)$, to be the number (when it exists), with the property that, for any $\epsilon > 0$, there is a neighborhood, U, of t such that*

$$\Big|[x^{\sigma}(t) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\Big| \leq \epsilon\Big|\sigma(t) - s\Big|,$$

for all $s \in U$. The second delta derivative *of $x(t)$ is define d by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t)$. If $F^{\Delta}(t) = h(t)$, then define the* integral *by*

$$\int_a^t h(s)\Delta s = F(t) - F(a).$$

Throughout this paper, we assume T is a closed subset of R with $0, 1 \in T$.

Definition 1.3 *Define the closed interval, $[0, 1]$, in T by*

$$[0, 1] = \{t \in T \mid 0 \leq t \leq 1\}.$$

Other closed, open, and half-open intervals in T are similarly defined.

In this paper, we are concerned with determining values of λ for which there exist positive solutions (u, v) of the system of differential equations on a measure chain, for $t \in [0, 1]$,

$$\begin{aligned} u^{\Delta\Delta}(t) + \lambda a(t) f(u^{\sigma}(t), v^{\sigma}(t)) &= 0, \\ v^{\Delta\Delta}(t) + \lambda b(t) g(u^{\sigma}(t), v^{\sigma}(t)) &= 0, \end{aligned} \tag{1}$$

satisfying the boundary conditions,

$$\begin{aligned} u(0) &= 0 = u^{\sigma}(1), \\ v(0) &= 0 = v^{\sigma}(1) \end{aligned} \tag{2}$$

where

(H$_1$) $f, g : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous, $f(x,y)$ is increasing in x for any fixed y, and $g(x,y)$ is increasing in y for any fixed x.

(H$_2$) $a, b : [0,\sigma(1)] \to (0,\infty)$ are continuous, and do not vanish identically on any closed subinterval of $[0,\sigma(1)]$.

(H$_3$) If $\|(x,y)\| > 0$, then $\|(f(x,y), g(x,y))\| > 0$.

(H$_4$) $\lim_{\|(x,y)\|\to 0} \frac{f(x,y)}{\|(x,y)\|} = \lim_{\|(x,y)\|\to 0} \frac{g(x,y)}{\|(x,y)\|} = 0. =$

(H$_5$) Either

(i) $\lim_{\|(x,y)\|\to\infty} \frac{f(x,y)}{\|(x,y)\|} = \infty$ or $\lim_{\|(x,y)\|\to\infty} \frac{f(x,y)}{\|(x,y)\|} = \infty$

or

(ii) $\lim_{\|x\|\to\infty} \frac{f(x,y)}{\|x\|} = \infty$ and $\lim_{\|y\|\to\infty} \frac{g(x,y)}{\|y\|} = \infty$ with both limits being uniform in the other variable,

or

(iii) $\lim_{\|y\|\to\infty} \frac{f(x,y)}{\|y\|} = \infty$ and $\lim_{\|x\|\to\infty} \frac{g(x,y)}{\|x\|} = \infty$ with both limits being uniform in the other variable.

We remark that by the solution, (u,v) of (1), (2), we mean $u, v : [0,\sigma^2(1)] \to R$, u, v satisfy (1), (2). We further remark that, if (u,v) is a nonnegative solution of (1), (2), then $u^{\Delta\Delta}(t) \le 0$ and $v^{\Delta\Delta}(t) \le 0$ on $[0,1]$, and we will say u, v are *concave* on $[0,\sigma^2(1)]$. Our works are motived by some recent results by Fink, *et al.* [8], as well as two papers by Fink [6] and Fink and Gatica [7]. In [8] the authors deal with the case $n = 2$ in modeling the one-dimesional case of the Dirichlet problem for $\Delta u + \lambda y(x) f(u) = 0, x \in \Omega \in R^N$. These results were then extended in [7] to find positive solutions of systems of second order boundary value problems. We extend those results to differential equations on a measure chain. In section 2, we present some properties of a Green's function which plays the role of a kernel of a completely continuous integral operator associated with (1), (2). We also state a fixed point theorem which can be found in [9]. In Section 3, we give an appropriate Banach space and construct a cone on which we apply the fixed point theorem yielding solutions of (1), (2), for every $\lambda > 0$ under reasonable conditions.

2 Some preliminaries

In this section, we state the above mentioned fixed point theorem. We will apply this fixed point theorem in the next section to a completely continuous integral operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta\Delta}(t) = 0, \tag{3}$$

$$y(0) = 0 = y^{\sigma}(1). \tag{4}$$

Erbe and Peterson [5] have found

$$G(t,s) = \begin{cases} \frac{t(\sigma(1)-\sigma(s))}{\sigma(1)}, & t \leq s, \\ \frac{\sigma(s)(\sigma(1)-t)}{\sigma(1)}, & \sigma(s) \leq t, \end{cases} \tag{5}$$

from which

$$G(t,s) > 0, \quad (t,s) \in (0,\sigma(1)) \times (0,1), \tag{6}$$

$$G(t,s) \leq G(\sigma(s),s) = \frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)}, \ t \in [0,\sigma(1)], \ s \in [0,1], \tag{7}$$

and it is also shown in [5] that

$$G(t,s) \geq kG(\sigma(s),s) = k\frac{\sigma(s)(\sigma(1)-\sigma(s))}{\sigma(1)}, \ t \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}], \ s \in [0,1], \tag{8}$$

where $m = \min\{\frac{1}{4}, \frac{\sigma(1)}{4(\sigma(1)-\sigma(0))}\}$.

To establish eigenvalue intervals we employ the following fixed point theorem which can be found in Gatica and Smith [9].

Theorem 2.1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{K} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Suppose that $0 < r < R$ are real numbers. Let $\mathcal{D} = \{x \in \mathcal{K} : r \leq \|x\| \leq R\}$ and $T : \mathcal{D} \to \mathcal{K}$ be a compact continuous operator such that*

(i) $x \in \mathcal{D}, \|x\| = R, Tx = \lambda x \Longrightarrow \lambda \geq 1$;

(ii) $x \in \mathcal{D}, \|x\| = r, Tx = \lambda x \Longrightarrow \lambda \leq 1$;

(iii) $\inf_{\|x\|=R} \|Tx\| > 0$,

then T has a fixed point in $\mathcal{D}$.

3 Characterization of $S_p(f, g)$

In this section, we apply Theorem 2.1 to the eigenvalue problem (1), (2). The notation $S_p(f, g)$ denotes the set of values of $\lambda \geq 0$ for which (1), (2) has a nonnegative solution. Throughout this section, we assume $\sigma(1)$ is right dense so that $G(t, s) \geq 0$, for $t \in [0, \sigma^2(1)]$, $s \in [0, \sigma(1)]$.

Assume also throughout this section that $[0, \sigma(1)]$ is such that

$$\xi = \min\{t \in T \mid t \geq \frac{\sigma(1)}{4}\}, \text{ and}$$
$$\omega = \max\{t \in T \mid t \leq \frac{3\sigma(1)}{4}\}$$

both exist and satisfy,

$$\frac{\sigma(1)}{4} \leq \xi < \omega \leq \frac{3\sigma(1)}{4},$$

and if $\sigma(\omega) = 1$, also assume $\sigma(\omega) < \sigma(1)$. Next, let $\tau \in [\xi, \omega]$ be defined by

$$\int_{\xi}^{\omega} G(\tau, s)a(s)\Delta s = \max_{t \in [\xi,\omega]} \int_{\xi}^{\omega} G(t, s)a(s)\Delta s. \tag{9}$$

For our constructions, let $\mathcal{C}[0, \sigma^2(1)]$ be the set of all continuous functions on $[0, \sigma^2(1)]$ and endowed with norm $||x|| = \max_{t \in [0,\sigma^2(1)]} |x(t)|$ and $\mathcal{B} = \mathcal{C}[0, \sigma^2(1)] \times \mathcal{C}[0, \sigma^2(1)]$ with the maximum norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$. Then, define the cone $\mathcal{P} \subset \mathcal{C}[0, \sigma^2(1)]$

$$\mathcal{P} = \{x \in \mathcal{C}[0, \sigma^2(1)] \mid x(t) \geq 0 \text{ concave on } [0, \sigma^2(1)], \text{ and } x(t) \geq m\|x\|, \text{ for } t \in [\xi, \sigma(\omega)]\}.$$

It is clear that $\mathcal{P} \times \mathcal{P}$ is a normal cone in $\mathcal{B}$. Since $(u(t), v(t))$ is a solution of (1), (2), for a given λ, if and only if

$$(\lambda \int_0^{\sigma(1)} G(t, s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s, \lambda \int_0^{\sigma(1)} G(t, s)b(s)g(u^\sigma(s), v^\sigma(s))\Delta s) = (u(t), v(t)),$$

we define the operator $T : \mathcal{P} \times \mathcal{P} \to \mathcal{B}$ by

$$T(u,v)(t) = (\int_0^{\sigma(1)} G(t,s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s, \int_0^{\sigma(1)} G(t,s)b(s)g(u^\sigma(s), v^\sigma(s))\Delta s).$$

We seek a fixed point of λT in $\mathcal{P}\times\mathcal{P}$. We need to show that λT is a compact operator from $\mathcal{P}\times\mathcal{P}$ to $\mathcal{P}\times\mathcal{P}$. First, if $(u,v) \in \mathcal{P}\times\mathcal{P}$ and $\lambda T(u,v) = (\phi,\psi)$, then by the properties of the Green's function G. We have $\phi(t) \geq 0, \psi(t) \geq 0$ and $\phi^{\Delta\Delta}(t) = -f(u^\sigma(t), v^\sigma(t)) \leq 0, \psi^{\Delta\Delta}(t) = -g(u^\sigma(t), v^\sigma(t)) \leq 0$. Moreover, for $t \in [\xi, \sigma(\omega)]$,

$$\begin{aligned}
\phi(t) &= \lambda \int_0^{\sigma(1)} G(t,s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s \\
&\geq \lambda \int_0^{\sigma(1)} mG(\sigma(s),s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s \\
&= m \int_0^{\sigma(1)} \lambda G(\sigma(s),s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s \\
&= m\|\phi\|,
\end{aligned}$$

and, similarly,

$\psi(t) \geq m\|\psi\|$ for $t \in [\xi, \sigma(\omega)]$. Therefore $\lambda T : \mathcal{P}\times\mathcal{P} \to \mathcal{P}\times\mathcal{P}$. Moreover, λT is completely continuous by a typical application of the Ascoli-Arzela Theorem. For ease of notation in the following discussion, set

$$A = \int_0^{\sigma(1)} G(\sigma(s),s)a(s)\Delta s, \quad \text{and} \quad B = \int_0^{\sigma(1)} G(\sigma(s),s)b(s)\Delta s.$$

We are now in a position to show the $S_p(f,g)$ contains an interval with 0 as its left endpoint.

Theorem 3.1 *Assume that conditions (H_1), (H_2) are satisfied. Then there exists $\delta > 0$ such that $[0,\delta) \subseteq S_p(f,g)$.*

Proof: To prove this theorem by using the Schauder fixed point theorem, we show that for sufficiently small $\lambda > 0$, T maps a closed subset of $\mathcal{P}\times\mathcal{P}$ into itself. Indeed for an $R > 0$, we define

$$\mathcal{P}_R \times \mathcal{P}_R = \{(u,v) \in \mathcal{P}\times\mathcal{P} : \|(u,v)\| \leq R\}.$$

Then

$$\|\lambda T(u,v)\| = \max\{\lambda \int_0^{\sigma(1)} G(t,s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s, \lambda \int_0^{\sigma(1)} G(t,s)b(s)g(u^\sigma(s), v^\sigma(s))\Delta s\}$$

$$
\begin{aligned}
&\leq \lambda \max\{\int_0^{\sigma(1)} G(\sigma(s),s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s, \int_0^{\sigma(1)} G(\sigma(s),s)b(s)g(u^\sigma(s),v^\sigma(s))\Delta s\} \\
&\leq \lambda \max_{0\leq s\leq 1}\{f(u^\sigma(s),v^\sigma(s)), g(u^\sigma(s),v^\sigma(s))\}\cdot \max\{A,B\}. \\
&< R,
\end{aligned}
$$

for sufficently small λ. This completes the proof.

The following theorem shows that $S_p(f,g)$ is an interval.

Theorem 3.2 *If* (1), (2) *has a positive solution for* $\lambda_1 > 0$*, then it has a solution for all* $0 < \lambda < \lambda_1$.

Proof: Let $(u,v) \in \mathcal{P}\times\mathcal{P}$ be solution of (1), (2) corresponding to λ_1 and let $0<\lambda<\lambda_1$. Then

$$
\lambda T(u,v)(t) < \lambda_1 T(u,v)(t) = (u(t),v(t)), \quad t\in[0,\sigma^2(1)]
$$

and hence $\{(\lambda T)^n\}$ is monotone decreasing. The existence of the fixed point of λT follows immediately from the properties of compactness of T and normality of the cone $\mathcal{P}\times\mathcal{P}$.

We now state the location of λ whenever $\lambda \in S_p(f,g)$.

Theorem 3.3 *Assume that conditions* (H_1), (H_2) *are satisfied. Let* $\lambda\in S_p(f,g)$ *and* (u,v) *be the corresponding solution of* (1), (2)*. Then we have*

$$
\frac{\|u\|}{\int_0^{\sigma(1)} G(\sigma(s),s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s} \leq \lambda \leq \frac{\|u\|}{int_\xi^\omega G(\tau,s)a(s)f(m\|u\|,v^\sigma(s))\Delta s}, \textit{ and} \tag{10}
$$

$$
\frac{\|v\|}{\int_0^{\sigma(1)} G(\sigma(s),s)b(s)g(u^\sigma(s),v^\sigma(s))\Delta s} \leq \lambda \leq \frac{\|v\|}{\int_\xi^\omega G(\tau,s)a(s)g(u^\sigma(s),m\|v\|)\Delta s}. \tag{11}
$$

Proof: We show (10). The proof of (11) is entirely analogous. Since u is in $\mathcal{P}$, $u(t)\geq m\|u\|$ on $[\xi,\sigma(\omega)]$, we have

$$
\begin{aligned}
\|u\| \geq u(\tau) &= \lambda \int_0^{\sigma(1)} G(\tau,s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s \\
&\geq \lambda \int_\xi^\omega G(\tau,s)a(s)f(m\|u\|,v^\sigma(s))\Delta s.
\end{aligned}
$$

On the other hand, for some $c\in[0,\sigma(1)]$,

$$
\begin{aligned}
\|u\| = u(c) &= \lambda \int_0^{\sigma(1)} G(c,s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s \\
&\leq \lambda \int_0^{\sigma(1)} G(\sigma(s),s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s.
\end{aligned}
$$

Thus (10) follows.

Finally we shall give a sufficient condition such that $S_p(f,g) = [0,\infty)$.

Theorem 3.4 *Assume conditions (H_1)-(H_5) are satisfied. Then, $S_p(f,g) = [0,\infty)$.*

Proof: It suffices to show that for any $\lambda_0 > 0$, $\lambda_0 T(u,v) = (u,v)$ has a fixed point. Let $\epsilon > 0$ be such that $\epsilon\lambda_0 \max\{A,B\} \leq 1$. Assumption ($H_4$) implies that there exists $r > 0$ such that $\|(x,y)\| \leq r$ implies $\|(f(x,y), g(x,y))\| \leq \epsilon r$. Let $\|(u,v)\| = r$ be in $\mathcal{P}\times\mathcal{P}$. Suppose $\lambda_0 T(u,v) = \mu(u,v)$ for some $\mu > 0$. We now show that $\mu \leq 1$.

$$\begin{aligned}
\mu r &= \mu\|(u,v)\| \\
&= \|\lambda_0 T(u,v)\| \\
&= \lambda_0\|(\int_0^{\sigma(1)} G(t,s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s, \int_0^{\sigma(1)} G(t,s)b(s)g(u^\sigma(s),v^\sigma(s))\Delta s)\|. \\
&\leq \lambda_0 \max\{A,B\}\epsilon r.
\end{aligned}$$

Thus $\mu \leq 1$.

Let $\eta > 0$ be such that

$$\lambda_0\eta\min\{\int_\xi^\omega G(t,s)a(s)\Delta s, \int_\xi^\omega G(t,s)b(s)\Delta s\} > \frac{1}{m}.$$

By assumption (i) of (H_5), there exists an $R > 0$ such that $\|(x,y)\| \geq mR$ implies $f(x,y) \geq \eta mR$ or $g(x,y) \geq \eta mR$. Let $\|(u,v)\| = R$ and $\lambda_0 T(u,v) = \mu(u,v)$. Then we have

$$\begin{aligned}
\mu R &= \mu\|(u,v)\| \\
&= \|\lambda_0 T(u,v)\| \\
&= \lambda_0\|(\int_0^{\sigma(1)} G(t,s)a(s)f(u^\sigma(s),v^\sigma(s))\Delta s, \int_0^{\sigma(1)} G(t,s)b(s)g(u^\sigma(s),v^\sigma(s))\Delta s)\| \\
&\geq \lambda_0\eta mR\min\{\int_\xi^\omega G(t,s)a(s)\Delta s, \int_\xi^\omega G(t,s)b(s)\Delta s\} \\
&> R.
\end{aligned}$$

and hence $\mu > 1$.

If either (ii) or (iii) hold, a simple modification of the argument yields again $\mu > 1$.

Now we are almost ready to apply the fixed point theorem; the last step is to prove that

$$\inf_{\|(u,v)\|=R} \|\lambda_0 T(u,v)\| > 0.$$

With this in mind, recall that $(u,v) \in \mathcal{P} \times \mathcal{P}$ then

$$u(t) \geq m\|u\| \text{ and } v(t) \geq m\|v\|, \quad t \in [\xi, \sigma(\omega)].$$

Suppose the (i) of (H_5) holds. If $\|(u,v)\| = R$, for $t \in [\xi, \sigma(\omega)]$,

$$\begin{aligned} \|\lambda_0 T(u,v)\| &\geq \lambda_0 \|(\int_\xi^\omega G(t,s)a(s)f(u^\sigma(s), v^\sigma(s))\Delta s, \int_\xi^\omega G(t,s)b(s)g(u^\sigma(s), v^\sigma(s))\Delta s)\| \\ &\geq \lambda_0 \min_{R \geq \|(x,y)\| \geq mR} \|(f(x,y), g(x,y))\| \min_{t \in [\xi, \sigma(\omega)]} \|(\int_\xi^\omega G(t,s)a(s)\Delta s, \int_\xi^\omega G(t,s)b(s)\Delta s)\| \\ &> 0. \end{aligned}$$

If (ii) or (iii) of (H_5) hold it is obvious how to modify the argument. Thus $\lambda_0 T$ must have a nonozero fixed point in $\mathcal{P} \times \mathcal{P}$, proving that $\lambda_0 T(u,v) = (u,v)$ has a fixed point.

References

[1] R. P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.* **35**(1999), 3-22.

[2] B. Aulback and S. Hilger, Linear dynamic processes with inhomogeneous time scale, Nonlinear Dynamics and Quantum Dynamical Systems, *Math. Res.* **59**, Akademie Verlag, Berlin, 1990.

[3] L. H. Erbe and S. Hilger, Sturmian theory on measure chains, *Differential Equations Dynam. Systems* **1**(1993), 223-246.

[4] L. H. Erbe and A. Peterson, Green's functions and comparison theorems for differential equations on measure chains, *Dynam. Contin., Discrete Impuls. Systems*, **6**(1999), 121-137.

[5] L. H. Erbe and A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, reprint.

[6] A. M. Fink, The radial Laplacian Gel'fand problem, *Delay and Differentaial equations*(Ames, IA, 1991), 93-98, World Sci. Publishing, River Edge, NJ, 1992.

[7] A. M. Fink and J. A. Gatica, Positive solutions of a sysytem of second order boundary value problem, *J. Math. Anal. Appl.*, **180**(1993), 93-108.

[8] A. M. Fink, J. A. Gatica and G. E. Heranandez, Eigenvalues of generalized Gel'fand models, *Nonlin. Anal.*, **26**(1993), 145 3-1468.

[9] J. A. Gatica and H. L. Smith, Fixed point theorems in a cone woth applications, *J. Math. Anal. Appl.*, **61**(1997), 58-71.

[10] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, *Resultate Math.* **18**(1990) , 18-56.

[11] B. Kaymakcalan, V. Lakshmikantham and S. Sivasundaram, *Dynamical Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.