行政院國家科學委員會專題研究計畫　成果報告

一對可交換方陣所生成的代數

計畫類別：個別型計畫
計畫編號：NSC91-2115-M-032-003-
執行期間：91 年 08 月 01 日至 92 年 07 月 31 日
執行單位：淡江大學數學系

計畫主持人：陳功宇

計畫參與人員：劉鑠榮

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 92 年 11 月 25 日

行政院國家科學委員會專題研究計畫成果報告

一對可交換方陣所生成的代數

The algebra generated by commuting pairs of matrices

計畫編號: NSC91-2115-M-032-003

執行期限: 91 年 8 月 1 日至 92 年 7 月 31 日

主持人: 陳功宇　　　執行機構及單位名稱: 淡江大學數學系

計畫參與人員: 劉鑠榮　　　執行機構及單位名稱: 淡江大學數學系

中文摘要

我們研究一對可交換方陣所生成的代數何時其維度恰是方陣的階

關鍵詞: 可交換方陣, 循環矩陣.

Abstract

Let A,B be n×n complex matrices, Alg(A,B) be the algebra generated by A,B and identity matrix I. It is known that dim Alg(A,B)≤n, if AB=BA, We study that when dim Alg(A,B)=n for two commuting pairs of matrices A,B.

key words: commuting matrices, cyclic matrices, unispectral matrices.

1. Introduction

The algebra generated by two commuting matrices is, in particular, a vector space; what can it dimension be?

For matrices of order n over algebraically closed fields, it is known that the largest possible value of that dimension is equals to n. This dimension inequality was proved by Gerstenhaber [3], who made non-trivial use of some techniques of algebric geometry. In 1990, Barria and Halmos [1] offer another proof, which use elementary methods only and which, moreover, concretely exhibits a vector basis of the generated algebra. It suggests that we consider the problem when dim Alg(A,B)=n for two commuting pairs of matrices A and B. The problem is difficult , we only have some partial results.

A unispectral matrix is a matrix with only one eigenvalue. A cyclic matrix is a matrix that have a cyclic vector.

Every unispectral matrix is similar to a direct sum of Jordan bolck. A Jordan block of size n with λ, denoted by $J(n,\lambda)$, is a square matrix such as is exemplified by

$$J(n,\lambda)=\begin{bmatrix} \lambda & 1 & 0 & \Lambda & 0 \\ 0 & \lambda & 1 & O & M \\ M & 0 & \lambda & O & 0 \\ M & & O & O & 1 \\ 0 & \Lambda & \Lambda & 0 & \lambda \end{bmatrix}..$$

The largest number of cyclic direct summands that occurs in a unispectral direct summand of the Jordan form is called the height. Our result is the following

2. Main results

Theorem 1. Let A be a unispectral matrix with size $2m$ and height 2. If the degree of the minimal polynomial of A is m and BA=AB, then

$\dim A\lg(A,B) < 2m$ if and only if

B is unispectral and $A^{m-1}(\lambda - B) = 0$ for the eigenvalue of B.

Theorem 2. Let A= $J(m,\lambda) \oplus \lambda I_k$, where I_k is the identity matrix of size k.

Let $P = O \oplus I_k$ and BA=AB. Then

(a) $\dim A\lg(A,B) = m + k \Rightarrow$ $PBP : Range(P) \to Range(P)$ is cyclic.

(b) B is nonunispectral and

$PBP : Range(P) \to Range(P)$ is cyclic $\Rightarrow \dim A\lg(A,B) = m + k$.

The following example show that if B is unispectral, then Theorem 2.(b) is not true.

Example:

Let A= $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and C= $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then AB=BA and AC=CA.

It is easy to see that $\dim A\lg(A,B) = 4$ but $\dim A\lg(A,C) = 3$.

參考資料:

[1] J.Barrio and P.R.Halmos, Vector bases for two commuting matrices, LAMA 27(1990),147-157.

[2]M.Gerstenhaber, On dominance and varieties of commuting matrices, Annals of Math.73(1961),324-348.

[3]R.M.Guralnick, A note on commuting pairs of matrices, LAMA 31(1992),71-75.

[4]T.Laffey and S.Lazarus, Two-generated commutative matrix subalgebras, Linear

Alg. and Appl. 147(1991)249-273.
[5]A.Wadsworth,On commuting pairs of matrices,LAMA27(1990),159-162.