行政院國家科學委員會專題研究計劃成果報告

**************************************************

*　　完全正矩陣之研究　　*

*　　A Study of Completely Positive Matrices　　*

*　　*

*　　*

**************************************************

計劃類別：☑ 個別計劃　☐ 整合型計劃

計劃編號：NSC90-2115-M-032-004

執行期間：90 年 8 月 1 日至 91 年 7 月 31 日

個別型計劃：計劃主持人：譚必信
國科會助理：謝旻宜
國科會助理：洪劍能
國科會助理：林淑慧
共同主持人：

整合型計劃：總計劃主持人：
子計劃主持人：

註：整合型計劃總報告與子計劃成果報告請分開編印各成一冊，彙整一起繳送國科會。

處理方式：☑可立即對外提供參考

（請打∨）☐一年後可對外提供參考

☐兩年後可對外提供參考

（必要時，本會得展延發表時限）

執行單位：淡江大學數學系

中華民國　91 年 10 月 8 日

# 摘要

我們稱一 $n$ 階實對稱方陣 $A$ 爲完全正, 若 $A$ 可寫成 $BB^t$ 對某一 $n\times m$ (逐項) 非負矩陣 $B$, 其中 $B'$ 是指 $B$ 的轉置矩陣。在完全正矩陣定義中出現的 $m$ 的最小可能值我們稱之爲 $A$ 的完全正秩並記作 $\#(A)$。在本計劃我們證得若 $A$ 爲 5 階完全正矩陣, 且最少有一項爲零, 則 $\#(A) \leq 6$。

關鍵詞: 完全正矩陣、完全正秩、雙非負矩陣。

## Abstract

An $n$-by-$n$ real symmetric matrix $A$ is called completely positive if $A$ can be represented as $BB^t$ for some $n$-by-$m$ (entrywise) nonnegative matrix $B$, where we use $B^t$ to denote the transpose of $B$. The minimal value of $m$ in a $BB^t$ representation of a given completely positive matrix $A$ is called the completely positive rank of $A$ or simply the CP rank of $A$ and is denoted by $\#(A)$. In this project we prove that the CP rank of a 5-by-5 completely positive matrix which has at least one zero entry is at most 6.

Key words: Completely positive matrix, CP rank, doubly nonnegative matrix.

## 1. Motivation and Aims

An $n$-by-$n$ symmetric matrix $A$ is called completely positive if $A$ can be represented as $BB^t$ for some $n$-by-$m$ (entrywise) nonnegative matrix $B$, where we use $B^t$ to denote the transpose of $B$. Alternatively, a completely positive matrix can be defined as a matrix which can be written as $A = b_1b_1^t + \cdots + b_mb_m^t$, where each $b_i \in \mathbb{R}^n$ is a nonnegative vector. Then $b_i$'s correspond to the columns of $B$ in the original definition. The minimal value of $m$ in a $BB^t$ representation (or the corresponding rank one representation) of a given completely positive matrix $A$ is called the completely positive rank or simply the CP rank of $A$ and is denoted by $\#(A)$. The main aim of this project is to treat the following two well-known open problems on completely positive matrices:

**Problem 1.** Determine which doubly nonnegative matrices are completely positive.

**Problem 2.** Given a positive integer $n$, determine the value of $\max\{\#(A) : A \text{ is } n\text{-by-}n, \text{ completely positive}\}$.

Below is the historical background of the above problems.

It is not difficult to see that an $n$-by-$n$ real symmetric matrix $A$ is completely positive if and only if the corresponding quadratic form $Q(x) = x^tAx$ can be rewritten as $Q = L_1^2 + \cdots + L_m^2$, where $L_1, \ldots, L_m$ are nonnegative linear functionals on $x = (x_1, \ldots, x_m)^t$. In the 1960's, P.H. Diananda, M. Hall and M. Newman (see Ref. [10], [15]) began the study of completely positive matrices from the point of view of quadratic forms. Their results imply that if $A$ is $n$-by-$n$, doubly nonnegative (i.e., nonnegative and positive semidefinite), and $n \leq 4$, then we can always write $A$ as $BB^t$, where $B$ is an $n$-by-$n$, nonnegative matrix; in other words, $A$ is completely positive. Hall and Newman [15] also gave an example to illustrate that for $n \geq 5$, an $n$-by-$n$ doubly nonnegative matrix need not be completely positive. Moreover, they also proved that if $A$ is $n$-by-$n$ completely positive, then it is always possible to write $A$ in the form $BB^t$, where $B$ is $n \times m$, nonnegative, with $m < 2^n$. They raised the question of determining the smallest such $m$ when $n$ is fixed. This is Problem 2 of our project.

Probably, completely positive matrices first found their applications in block design (see Hall [14]). In [13] Gray and Wilson also pointed out that completely positive matrices can find applications in statistics and in a mathematics model for energy demand. P. Diaconis [9] also pointed out that completely positive matrices also arose in the study of the exchangeable probability distribution of finite sample spaces. Because of the needs from applications, clearly Problem 1 of our project is also a fundamental question.

The study of completely positive matrices has attracted the attention of many famous mathematicians, including C.R. Johnson, T.J. Laffey, M. Hall, R. Loewy, M. Neumann, A. Berman, D. Hershkowtiz, R. Grone, T. Ando, etc. So far, more than twenty papers have appeared. (Please refer to our reference list at the end.) In the eighties, completely positive matrices has been a "hot" topic. The climax occurred in the early nineties, when under the joint work several people (including Berman, Grone, Hershkowitz, Kogan, etc.), the problem of characterizing completely positive graphs was completely solved. We call a (undirected) graph $G$ completely positive if every doubly nonnegative matrix with graph equal to $G$ is completely positive. They proved that a graph is completely positive if and only if it does not contain an odd cycle of length 5 or more.

In this project we have mainly focused ourselves on the following conjecture, made by Drew, Johnson and Loewy [12] in 1994. Clearly, if this conjecture is true, then Problem 2 is completely settled.

**Conjecture.** If $A$ is $n$-by-$n$ completely positive, $n \geq 4$, then $\#(A) \leq [n^2/4]$.

As evidence for the conjecture, they proved in [12] that if $A$ is $n$-by-$n$ completely positive, $n \geq 4$, and the graph of $A$ is triangle-free (i.e., it contains no cycle of length 3), then $\#(A) \leq [n^2/4]$.

Two years later, Drew and Johnson [11] showed that the conjecture is true for every completely positive matrix whose graph is completely positive. More recently, Berman and Shaked-Monderer [7] proved that the conjecture is also true for every completely positive matrix $A$ for which the comparison matrix $M(A)$ is an $M$-matrix.

It is also worthwhile to mention the following two related results obtained in [12]:

**Theorem A.** *If $A$ is a symmetric nonnegative matrix and $G(A)$ is triangle-free, then $A$ is completely positive if and only if $M(A)$ is an $M$-matrix.*

**Theorem B.** *If $A$ is a symmetric nonnegative matrix, $G(A)$ is connected and $M(A)$ is an $M$-matrix, then $A$ is completely positive and*

$$\#(A) \leq \max\{|V(G(A))|, \ |E(G(A))|\},$$

*where $E(G(A))$ (respectively, $V(G(A))$) denotes the edge set (respectively, vertex set) of $G(A)$, and for a set $S$ we use $|S|$ to denote its cardinality.*

## 2. Results and Discussions

In this project we obtain the following main result, as new supporting evidence for the above Conjecture:

**Theorem.** *If $A$ is a 5-by-5 completely positive matrix which has at least one zero entry, then $\#(A) \leq 6$.*

The proof of the Theorem takes more than 11 pages. Below is a sketch of the ideas of the proof:

Let $A = (a_{ij}) \in \mathrm{CP}_5$, the set of 5-by-5 completely positive matrices. We want to prove that if $A$ has at least one zero entry, then $\#(A) \leq 6$. Since the property of being CP and also the CP rank are both invariant under permutation similarity, we may assume hereafter that $a_{12} = 0$.

We denote by $\mathbb{R}^n_+$ the set of all nonnegative vectors of $\mathbb{R}^n$.

We start with any rank 1 CP representation of $A$, say, $A = \sum_{j=1}^{m} b_j b_j^t$, where $b_j \in \mathbb{R}^5_+$, $j = 1, \ldots, m$. Note that for each $j$, $1 \leq j \leq m$, either the first or the second component of $b_j$ is zero. Let $\Lambda_1 = \{j$: the second component of $b_j$ is $0\}$, and let $\Lambda_2 = \{1, 2, \ldots, m\} \backslash \Lambda_1$. Also let $A_1 = \sum_{j \in \Lambda_1} b_j b_j^t$ and $A_2 = \sum_{j \in \Lambda_2} b_j b_j^t$. Then we obtain a decomposition of $A$:

$$A = A_1 + A_2, \text{where } A_i \text{ is CP, } i = 1, 2, \text{ and the second (respectively, first) row of } A_1 \text{ (respectively, } A_2\text{) is zero.} \quad (1)$$

Since the second row and column of $A_1$ are zero, $A_1$ is permutationally similar to the direct sum of a 4-by-4 CP matrix and the 1-by-1 zero matrix. But the CP rank of a 4-by-4 CP matrix is at most 4, so it follows that we have $\#(A_1) \leq 4$. For a similar reason, we also have $\#(A_2) \leq 4$. Thus, by a simple argument we have $\#(A) \leq 8$, but this is still far from our target.

We shall make use of the following observations.

By the *support* of a vector $x$, denoted by $\mathrm{supp}(x)$, we mean the set of indices associated with the nonzero components of $x$.

**Observation 1.** Let $u,\ v \in \mathbb{R}^n_+$. If $\mathrm{supp}(v) \subseteq \mathrm{supp}(u)$, then there exist $\tilde{u},\ \tilde{v} \in \mathbb{R}^n_+$, satisfying $uu^t + vv^t = \tilde{u}\tilde{u}^t + \tilde{v}\tilde{v}^t$, such that $\mathrm{supp}(u) = \mathrm{supp}(\tilde{u})$, and for some permutation matrix $P$, the vectors $Pu$, $P\tilde{u}$, $Pv$ and $P\tilde{v}$ can be partitioned identically so that they have the following sign patterns:

$$Pu = \left[\begin{array}{c} + \\ \cdot \\ \cdot \\ \cdot \\ + \\ \hline + \\ \vdots \\ + \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \quad Pv = \left[\begin{array}{c} + \\ \cdot \\ \cdot \\ \cdot \\ + \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \quad P\tilde{u} = \left[\begin{array}{c} + \\ \cdot \\ \cdot \\ \cdot \\ + \\ \hline + \\ \vdots \\ + \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \quad \text{and } P\tilde{v} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ + \\ \vdots \\ + \\ \hline + \\ \vdots \\ + \\ \hline 0 \\ \vdots \\ 0 \end{array}\right],$$

where the third group (in the partition) may be empty, the second group is empty if $\mathrm{supp}(u) = \mathrm{supp}(v)$, and the first group of $P\tilde{v}$ has at least one 0 and may contain all 0's.

This can be done by applying a procedure, which first appeared in Hall [14, proof of Lemma 16.2.1] in the context of a completely positive quadratic form for the special case when $u$ and $v$ have the same support. We shall refer to it as the *generalized Hall procedure*, or simply the *GH procedure*.

Suppose $u = (u_1, \ldots, u_n)^t$ and $v = (v_1, \ldots, v_n)^t$. Then for any real number $\theta$, we have

$$uu^t + vv^t = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_n & v_n \end{bmatrix} R_\theta^t R_\theta \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix},$$

where $R_\theta$ denotes the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Since $\mathrm{supp}(v) \subseteq \mathrm{supp}(u)$, each of the vectors $(u_1, v_1)^t, \ldots, (u_n, v_n)^t$ is of one of the following forms $(+,+)^t$, $(+,0)^t$ or $(0,0)^t$. The action of $R_\theta$ on these vectors is to rotate all of them counterclockwise by the same angle $\theta$. (In case $\mathrm{supp}(u) = \mathrm{supp}(v)$, we may also use a clockwise rotation.) We increase $\theta$ from zero gradually until it first happens that one (or more) of the vectors of the form $(+,+)^t$ becomes one of the form $(0,+)^t$. Then the resulting vectors all remain nonnegative, and vectors of the form $(+,0)^t$ now take the form $(+,+)^t$. Denote the corresponding value of $\theta$ by $\theta_0$, and let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^t$ and $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)^t$ be the vectors given by:

$$R_{\theta_0} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \cdots & \tilde{v}_n \\ \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_n \end{bmatrix}.$$

It is easy to check that we have $uu^t + vv^t = \tilde{u}\tilde{u}^t + \tilde{v}\tilde{v}^t$, and the vectors $\tilde{u}$, $\tilde{v}$ have the desired sign patterns.

**Observation 2.** Let $A \in \mathrm{CP}_n$ and let $S \in \mathbb{R}^{n,n}$ be such that $S$ is invertible and $S^{-1} \geq 0$. Let $B = SAS^t$. Then, if $B \in \mathrm{CP}_n$ we have $\#(A) \leq \#(B)$.

This is, of course, an obvious known observation (in which it suffices to assume $A \in \mathbb{R}^{n,n}$). It will be used several times in this paper. As in [1, proof of Theorem 2.6] we shall use $S$ that describes an elementary operation, or more precisely $S$ will have the form $S = I_n - aE_{ij}$, where $a > 0$, $i \neq j$, and $E_{ij}$ denotes the $n$-by-$n$ matrix with 1 at its $(i,j)$ position and 0 elsewhere. It is easy to see that for such $S$, $S^{-1}$ exists and is nonnegative.

**Observation 3.** Let $A = (a_{ij}) \in \mathrm{CP}_5$ be such that $a_{12} = 0$. Consider a decomposition of $A$ as given by (1). Suppose that for some $i$, $3 \leq i \leq 5$, and some $a > 0$, the matrix $S = I_5 - aE_{i2}$ satisfies $SA_2S^t \in \mathrm{CP}_5$. Then $SAS^t \in \mathrm{CP}_5$. (If we replace $E_{i2}$ and $A_2$ respectively by $E_{i1}$ and $A_1$, the assertion still holds.)

This is in fact quite obvious. The congruence we perform amounts to multiplying row 2 by $-a$ and adding it to row $i$, and doing the corresponding column operation. This does not change $A_1$ at all, so we have $SAS^t = SA_1S^t + SA_2S^t = A_1 + SA_2S^t$. Thus, $SAS^t$ is a sum of two matrices in $\mathrm{CP}_5$.

We shall also need the following lemmas:

**Lemma 1.** *Let $B = (b_{ij}) \in \mathrm{DNN}_n$, $n \geq 2$. Suppose that for some $r \neq s$, $1 \leq r,\ s \leq n$, the support of row $r$ is nonempty and is a subset of the support of row $s$. Then there exists $\alpha > 0$ such that for $S = I_n - \alpha E_{sr}$, $\widetilde{B} = SBS^t \in \mathrm{DNN}_n$ and has the property that the support of its row $s$ is a proper subset of that of the corresponding row of $B$.*

**Lemma 2.** *Let*

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & 0 & 0 \\ p_{13} & 0 & p_{33} & p_{34} \\ p_{14} & 0 & p_{34} & p_{44} \end{bmatrix}$$

*be a* rank 3 DNN *matrix with $p_{12} > 0$. Then $\#(P) = 3$.*

To prove our theorem, we start with the decomposition (1), and use Observation 3 repeatedly and systematically to obtain more 0's in the (transformed) matrices $A_1$ and $A_2$. In view of Observation 2, if we can show at the end of this process that the transformed matrix $A$ has CP rank 6 or less, then so does the original $A$. The remaining argument will take more than seven pages. We omit the details.

## 3. Self-evaluation of Performance

This project has been carried out pretty well. As described above, we successfully proved that the Conjecture due to Drew, Johnson and Loewy is true also for the case when $A$ is a 5-by-5 completely positive matrix with at least one zero entry. If it were true that every 5-by-5 doubly nonnegative matrix was completely positive, then by applying a suitable congruence (see our Lemma 1), we could reduce a 5-by-5 completely positive matrix all of whose entries are nonzero to a doubly nonnegative, and hence a completely positive matrix with at least one zero entry. Then by our main Theorem (and Observation 2), it would follow that the conjecture was true for all $A \in \mathrm{CP}_5$. Unfortunately, for $n \geq 5$, not every $n$-by-$n$ doubly nonnegative matrix is completely positive. So we have not yet fully verified the conjecture for the case when $n = 5$.

The work done in this project is included in a joint paper with Raphael Loewy [20] and will appear in Linear Algebra and Its Applications.

Recently, we learned that Francesco Barioli has found a counter-example of the above Conjecture for the case $n = 6$. So Problem 2 of our project remains an open problem.

Because of the lack of time, we have not done much towards Problem 1.

## REFERENCES

[1] T. Ando, *Completely positive matrices*, Lecture Notes, Sapporo, Japan, 1991.

[2] A. Berman, Complete positivity, *Linear Algebra Appl.* **107** (1988), 57–63.

[3] A. Berman, Completely positive graphs, in: *Combinatorial and Graph-Theoretical Problems in Linear Algebra*, The IMA Volumes in Mathematics and Its Applications, Volume 50, Springer-Verlag, NY, 1993, pp. 229–233.

[4] A. Berman and R. Grone, Bipartite completely positive matrices, *Proc. Cambridge Philos. Soc.* **103** (1988), 269–276.

[5] A. Berman and D. Hershkowitz, Combinatorial results on completely positive matrices, *Linear Algebra Appl.* **95** (1987), 111–125.

[6] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Revised reprint of the 1979 original, Classics in Applied Mathematics, 9, SIAM, Philadelphia, 1994.

[7] A. Berman and N. Shaked-Monderer, Remarks on completely positive matrices, *Linear and Multilinear Algebra* **44** (1998), 149–163.

[8] R.W. Cottle, G.J. Habetler and C.E. Lemke, On classes of copositive matrices, *Linear Algebra Appl.* **3** (1970), 295–310.

[9] P. Diaconis, Talk at IMA Workshop, November, 1993.

[10] P.H. Diananda, On non-negative forms in real variables some or all of which are non-negative, *Proc. Cambridge Philos. Soc.* **58** (1962), 17–25.

[11] J.H. Drew and C.R. Johnson, The no long odd cycle theorem for completely positive matrices, *The IMA Vol. Math. Appl.* **76**, 103–115, Springer, New York, 1996.

[12] J.H. Drew, C.R. Johnson and R. Loewy, Completely positive matrices associated with M-matrices, *Linear and Multilinear Algebra* **37** (1994), 303–310.

[13] L.J. Gray and D.G. Wilson, Nonnegative factorization of positive semidefinite nonnegative matrices, *Linear Algebra Appl.* **31** (1980), 119–127.

[14] M. Hall Jr., *Combinatorial Theory*, Blaisdell, Lexington, 1967; 2nd ed., 1986.

[15] M. Hall, Jr. and M. Newman, Copositive and completely positive quadratic forms, *Proc. Cambridge Philos. Soc.* **59** (1963), 329–339.

[16] J. Hannah and T.J. Laffey, Nonnegative factorization of completely positive matrices, *Linear Algebra Appl.* **55** (1983), 1–9.

[17] M. Kaykobad, On nonnegative factorization of matrices, *Linear Algebra Appl.* **96** (1987), 27–33.

[18] M. Kogan and A. Berman, Characterization of completely positive graphs, *Discrete Math.* **114** (1993), 297–304.

[19] C.M. Lau and T.L. Markham, Square triangular factorizations of completely positive matrices, *J. Industrial Math. Soc.* **28** (1978), 15–24.

[20] R. Loewy and B.S. Tam, CP rank of completely positive matrices of order five, accepted by *Linear Algebra Appl.*

[21] T.L. Markham, Factorization of completely positive matrices, *Proc. Cambridge Philos. Soc.* **69** (1971), 53–58.

[22] J.E. Maxfield and H. Minc, On the equation $X'X = A$, *Proc. Edinburgh Math. Soc.* **13** (1962), 125–129.

[23] X.D. Zhang, Laplacian matrices and completely positive matrices (in Chinese), Ph.D. thesis, University of Science and Technology of China, 1998.

# 出席國際會議報告

會議名稱：國際組合矩陣理論會議

會議地點：韓國浦項市

會議時間：91 年 1 月 14 日--17 日

報 告 人：淡江大學數學系　譚必信

撰寫日期：91 年 1 月 30 日

國際組合矩陣理論會議於本年 1 月 14 日至 17 日在韓國浦項市舉行。該會議是由浦項科技大學的組合及計算數學中心（$Com^2MaC$）主辦，並獲得韓國的科學與工程基金及科學與科技部的贊助。參加者來自世界各地，共約 70 人，其中以來自地主國（33 人）及美國（10 人）的佔多數。據我瞭解，國際組合矩陣理論會議這是第一次舉辦。韓國數學家在這個領域頗活躍，所以這次是由他們主辦。與會的華人或華裔數學家也不少，包括：邵嘉裕、劉柏廉、李喬、沈建、詹興致、陳安及本人。

在兩個全天及兩個半天的會議中，總共安排了 6 個一小時的大會演講（講者包括：Richard Brualdi、Steve Kirkland、Arnold Kraeuter、Bryan Shader、邵嘉裕及沈建）、11 個 40 分鐘的邀請演講及 27 個 25 分鐘的報告。40 分鐘及 25 分鐘的演講是分兩組平行進行。

組合矩陣理論是本人新的研究興趣。最近我跟以色列理工大學的 Raphael Loewy 教授合作，研究錐體正變變換的圖形，特別是有關本

原錐體正變換（*K*-primitive operator）方面。這次會議跟（非負的）本原矩陣有關的演講有 4 個，包括：劉柏廉的“Generalized exponents of Boolean matrices”、Bryan Shader 的“Exponents of tuples of nonnegative matrices”、LeRoy Beasley 的“*k*-primitive matrices”及 Coral Neal 的“2-primitivity of tournaments”，本人獲益良多。另外，R.A. Brualdi 的演講“Linear preservers and diagonal hypergraphs”、詹興致的“On permutations of matrix entries”、Steve Kirkland 的“An approach to algebraic connectivity via nonnegative matrices”等等，本人都甚感興趣。

本人的演講是安排在會議最後一天上午的平行小組一，講題爲“Graphs for cone-preserving maps revisited”，是 40 分鐘的報告。在同一時間 Willem Haemers 在另一組報告熱門題目“Which graphs are determined by their spectrum ?”可能是這個原因，我的聽眾只有六、七位。演講畢，邵嘉裕及詹興致都表示對我的工作很感興趣，並向我索取一些資料。

總括來說，我對韓國人舉辦這次會議的印象是很不錯，他們派出很多人力，工作很認真，細節都有注意到。

攜回資料：會議議程及摘要一本。

# GRAPHS FOR CONE-PRESERVING MAPS REVISITED

**Raphael Loewy**
Department of Mathematics
Technion
Haifa 32000, Israel

and

**Bit-Shun Tam**
Department of Mathematics
Tamkang University
Tamsui, Taiwan 25137, R.O.C.

Abstract. Let $K$ be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$ and let $A$ be an $n$-by-$n$ real matrix that satisfies $AK \subseteq K$. Let $(\mathcal{E}, \mathcal{P}(A, K))$ denote the digraph with vertex set $\mathcal{E}$ consisting of the extreme rays of $K$ such that $(E_1, E_2)$ is an arc if and only if $E_2 \subseteq \Phi(AE_1)$, where $\Phi(S)$ denotes the face of $K$ generated by $S$. We show that the $K$-irreducibility or $K$-primitivity of $A$ is completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ together with a knowledge of when a given finite collection $\{E_1, \ldots, E_j\}$ of extreme rays satisfies $E_1 \vee \cdots \vee E_j = K$. We treat the problem of determining the exponent of a $K$-primitive matrix $A$. We also touch upon the question of when a given digraph $G$ allows (or requires) the existence of a ($K$-irreducible or $K$-primitive) matrix $A \in \pi(K)$ such that $(\mathcal{E}, \mathcal{P}(A, K)) = G$, with $K$ being fixed or not fixed. Some open questions are also posed.

## 1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

We assume a basic knowledge of cones. For references, see [Bar 2].

Let $K$ be a proper cone in $\mathbb{R}^n$. Let $\pi(K)$ denote the set of all $n$-by-$n$ real matrices $A$ that satisfy $AK \subseteq K$. In [B–T] and [T–B], for any $A \in \pi(K)$, four directed graphs associated with $A$ are introduced by Barker and Tam as follow: Let $\mathcal{E}$ denote the set of all extreme rays of $K$, and let $\mathcal{F}'$ denote the set of all nontrivial faces of $K$. If $F, G \in \mathcal{F}'$, we say there is a $\mathcal{P}$-arc from $F$ to $G$ if $\Phi(AF) \supseteq G$, and an $\mathcal{I}$-arc from $F$ to $G$ if $\Phi((I+A)F) \supseteq G$, where $\Phi(S)$ denotes the face of $K$ generated by the subset $S$. Let $\mathcal{P}$ and $\mathcal{I}$ denote the set of $\mathcal{P}$-arcs and $\mathcal{I}$-arcs respectively. Then $(\mathcal{F}', \mathcal{P})$ (respectively $(\mathcal{F}', \mathcal{I})$) denotes the directed graph with vertex set $\mathcal{F}'$ and arc set $\mathcal{P}$ (respectively, $\mathcal{I}$). If necessary, we write $\mathcal{F}'(K)$, $\mathcal{P}(A)$, $\mathcal{P}(A, K)$ etc. to indicate the dependence on $K$, on $A$, and on $A$ and $K$ respectively. The digraphs $(\mathcal{E}, \mathcal{P})$ and $(\mathcal{E}, \mathcal{I})$ are defined in a similar manner. If $K$ equals the nonnegative orthant $\mathbb{R}^n_+$, then $\mathcal{E} = \{\Phi(e_1), \ldots, \Phi(e_n)\}$, where $e_1, \ldots, e_n$ are the standard unit vectors of $\mathbb{R}^n$. In this case, $(\Phi(e_i), \Phi(e_j))$ is a $\mathcal{P}$-arc if and only if $a_{ji} > 0$; hence, we can identify $(\mathcal{E}, \mathcal{P})$ with $G(A^T)$, where $G(A)$ is the usual digraph associated with the square matrix $A$.

A matrix $A \in \pi(K)$ is said to be *$K$-irreducible* if $A$ leaves no nontrivial face of $K$ invariant, $A$ is *$K$-positive* if $A(K \backslash \{0\}) \subseteq \text{int } K$ and is *$K$-primitive* if there is a positive integer $p$ such that $A^p$ is $K$-positive. If $A$ is $K$-primitive, then the smallest positive integer $p$ for which $A^p$ is $K$-positive is called the *exponent* of $A$ and is denoted by $\gamma(A)$.

It is well-known (see, for instance, [B–R]) that a nonnegative matrix $A$ is irreducible if and only if its digraph $G(A)$ is strongly connected; $A$ is primitive if and only if $G(A)$ is strongly connected and the greatest common divisor of the lengths of its circuits equals 1. In contrast, in [B–T] it is proved that, for any $A \in \pi(K)$, the $K$-irreducibility (respectively, $K$-primitivity) of $A$ is equivalent to the strong connectedness of the digraph $(\mathcal{F}', \mathcal{I})$ (respectively, $(\mathcal{F}', \mathcal{P})$). The following diagram summarizes the connections between the strong connectedness of the four digraphs:

$$\begin{array}{ccc} (\mathcal{E}, \mathcal{P}) \text{ is strongly connected} & & (\mathcal{F}', \mathcal{P}) \text{ is strongly connected} \\ \Downarrow & & \Downarrow \\ (\mathcal{E}, \mathcal{I}) \text{ is strongly connected} & \Longrightarrow & (\mathcal{F}', \mathcal{I}) \text{ is strongly connected} \end{array}$$

In the subsequent paper [T–B], it is proved that, loosely speaking, the phenomenon of irreducibility of operators being determined by the extreme rays alone is characteristic of simplicial cones. More specifically, the following are obtained:

**Theorem A.** *$K$ is simplicial if for any $A \in \pi(K)$, $(\mathcal{E}, \mathcal{P}(A))$ is strongly connected whenever $A$ is $K$-irreducible.*

**Theorem B.** *Suppose that $K$ is a 2-neighborly proper cone. Then $K$ is simplicial if, for any $A \in \pi(K)$, $(\mathcal{E}, \mathcal{I}(A))$ is strongly connected whenever $A$ is $K$-irreducible.*

Here we call a proper cone $K$ *2-neighborly* if, for any two extreme vectors $x_1, x_2 \in K$, $x_1 + x_2 \in \partial K$. Actually, Theorem B can be strengthened slightly by replacing "$K$-irreducible" by "$K$-primitive"; the proof is just a minor modification of the original proof for Theorem B as given in [T–B].

It is well-known that if $K$ is a proper cone [respectively, polyhedral (proper) cone] in $\mathbb{R}^n$, then $\pi(K)$ is a proper cone [respectively, polyhedral cone] in the space of $n$-by-$n$ real matrices (see [S–V] or [Tam 3]).

In [Niu] Niu considered the exponents of $K$-primitive matrices on a polyhedral cone $K$. We summarize his results in the following:

**Theorem C.** *Let $K$ be a polyhedral cone, and let $A, B \in \pi(K)$.*

(i) $(\mathcal{E}, \mathcal{P}(A)) = (\mathcal{E}, \mathcal{P}(B))$ *if and only if* $\Phi(A) = \Phi(B)$.

(ii) *If $(\mathcal{E}, \mathcal{P}(A))$ is a subdigraph of $(\mathcal{E}, \mathcal{P}(B))$ and if $A$ is $K$-primitive, then so is $B$ and we have $\gamma(B) \leq \gamma(A)$.*

(iii) *If $\Phi(A) = \Phi(B)$, then $A$ and $B$ are both $K$-primitive or both not $K$-primitive, and if they are, then $\gamma(A) = \gamma(B)$.*

**Theorem D.** *Let $K$ be a polyhedral cone with $m$ extreme rays, and let $A$ be $K$-primitive. If the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected and $s$ is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$, then $\gamma(A) \leq m + s(m-2)$.*

In Theorem D, by choosing $K = \mathbb{R}^n_+$ one obtains $n + s(n-2)$, the classical upper bound for $\gamma(A)$ due to Dulmage and Mendelsohn [D–M]. By setting $s = n-1$, one also recovers the exact general upper bound $(n-1)^2 + 1$, due to Wielandt [Wie] for exponents of primitive matrices of order $n$.

In this paper we continue the study of (di)graphs of a cone-preserving map and their connections with $K$-irreducibility and $K$-primitivity.

For an $n$-by-$n$ nonnegative matrix $A$, clearly the digraph $(\mathcal{E}, \mathcal{P}(A))$ and the relative position of $A$ in $\pi(\mathbb{R}^n_+)$ (or equivalently, the face $\Phi(A)$) determines each other (as they are each determined by the zero-nonzero pattern of $A$). Theorem C(i) tells us that the same is true when the underlying cone $K$ is polyhedral. However, when $K$ is a general proper cone, the situation is more complicated, as we will show in this paper.

For a proper cone $K$, we use $\mathrm{cl}_K$ to denote the composite map $d_{K^*} \circ d_K$, where $d_K$ denotes the duality operator of $K$. For the necessary background knowledge of the duality operators (of $K$ and of $\pi(K)$), we refer the reader to [Tam 2,3]. Putting it in another way, the set $\mathrm{cl}_K(\Phi(x))$ is equal to the smallest exposed face of $K$ generated by $x$; $\mathrm{cl}_K(\Phi(x))$ equals $K$ if $x \in \mathrm{int}\, K$ and equals the intersection of $K$ with all the hyperplanes which support $K$ at $x$ if $x \in \partial K$.

For convenience, we also use Ext $K$ to denote the set of nonzero extreme vectors of $K$.

**Theorem 1.** *Let $K$ be a proper cone, and let $A, B \in \pi(K)$. Consider the following conditions*:

(a) $\Phi(A) \subseteq \Phi(B)$.

(b) $(\mathcal{E}, \mathcal{P}(A))$ *is a subdigraph of* $(\mathcal{E}, \mathcal{P}(B))$.

(c) *For all* $x \in \mathrm{Ext}\ K$, $\Phi(Ax) \subseteq \Phi(Bx)$.

(d) *For all* $x \in K$, $\Phi(Ax) \subseteq \Phi(Bx)$.

(e) $(\mathcal{F}', \mathcal{P}(A))$ *is a subdigraph of* $(\mathcal{F}', \mathcal{P}(B))$.

(f) $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$.

*Conditions* (b)–(e) *are equivalent and they always imply condition* (f) *and are implied by condition* (e).

By a *simple face* of $\pi(K)$ we mean a face of the form $\pi_{F,G}$ for some faces $F$, $G$ of $K$, where $\pi_{F,G} = \{A \in \pi(K) : AF \subseteq G\}$ (see [Tam 3]).

By Theorem 1 we immediately obtain

**Corollary 1.** *Let $K$ be a proper cone, and let $A, B \in \pi(K)$. Consider the following conditions*:

(a) $\Phi(A) = \Phi(B)$

(b) $(\mathcal{E}, \mathcal{P}(A)) = (\mathcal{E}, \mathcal{P}(B))$.

(c) *For all* $x \in \mathrm{Ext}\ K$, $\Phi(Ax) = \Phi(Bx)$.

(d) *For all* $x \in K$, $\Phi(Ax) = \Phi(Bx)$.

(e) $(\mathcal{F}', \mathcal{P}(A)) = (\mathcal{F}', \mathcal{P}(B))$.

(f) $\mathrm{cl}_{\pi(K)}(\Phi(A)) = \mathrm{cl}_{\pi(K)}(\Phi(B))$.

*Conditions* (b)–(e) *are equivalent and they always imply condition* (f) *and are implied by condition* (a).

**Remark 1.** For the conditions (a)–(f) of Theorem 1, condition (a) implies condition (d) for all pairs of matrices $A, B \in \pi(K)$ if and only if every face of $\pi(K)$ can be written as an intersection of simple faces (see [Tam 3, Corollary 4.7]).

Among the four digraphs associated with $A$ in $\pi(K)$, the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is the simplest one, as it has the fewest vertices and arcs. Moreover, it reduces to the usual digraph (but with the direction of arcs reversed) in the nonnegative matrix case. It would be nice if one can capture the other three digraphs from $(\mathcal{E}, \mathcal{P}(A, K))$; then we need only work with this digraph. The equivalence of conditions (b) and (e) in Corollary 1 suggests that this should be feasible. Indeed, we need to have only a knowledge of the inclusion relations between the faces of $K$, i.e., to know $K$ up to combinatorial equivalence.

**Theorem 2.** *Suppose the digraph $(\mathcal{F}', \mathcal{P}(I, K))$, or equivalently the inclusion relation between the faces of $K$, is given. Then from the digraph $(\mathcal{F}', \mathcal{P}(A, K))$ one can determine the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and conversely. When either one of the two aforementioned digraphs is known, one can also determine the digraphs $(\mathcal{E}, \mathcal{I}(A, K))$ and $(\mathcal{F}', \mathcal{I}(A, K))$ (but not conversely).*

According to [Tam 3, Corollary 3.4], if $A, B \in \pi(K)$ satisfy $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$, and if $A$ is $K$-irreducible, then $B$ is also $K$-irreducible. For $K$-primitivity, we can prove a similar result. Making use of it, we can complete the result of Theorem C, parts(ii) and (iii).

**Theorem 3.** *Let $K$ be a proper cone, and let $A, B \in \pi(K)$.*

(i) *If $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$ and $A$ is $K$-primitive, then $B$ is also $K$-primitive and we have $\gamma(B) \leq \gamma(A)$.*

(ii) *If $\mathrm{cl}_{\pi(K)}(\Phi(A)) = \mathrm{cl}_{\pi(K)}(\Phi(B))$, then $A$ and $B$ are both $K$-primitive or both not $K$-primitive, and if they both are, then $\gamma(A) = \gamma(B)$.*

For a proper cone $K$, we say $K$ *has finite exponent* if the set of exponents of $K$-primitive matrices is bounded above; then we denote the maximum exponent by $\gamma(K)$. This concept has some connection with the known concept of critical exponent of a norm. In [B–L, p.66], an example of a norm in $\mathbb{R}^2$ for which the critical exponent does not exist is provided. Borrowing the example, one can readily construct a proper cone in $\mathbb{R}^3$ which does not have finite exponent. (See our Example 7.) However, if $K$ is a polyhedral cone, then $\pi(K)$ is also a polyhedral cone. As such, $\pi(K)$ has finitely many faces. By Theorem C(iii), $K$-primitive matrices that belong to the relative interior of the same face of $\pi(K)$ all have the same exponents. Hence, there are finitely many values for $\gamma(A)$, as $A$ runs through all $K$-primitive matrices. This establishes the following:

**Corollary 2.** *Every polyhedral cone has finite exponent.*

For simplicity, we call a strongly connected component of a digraph a *strong component.* We call a strong component *final* if there is no arc that issues from the strong component and enters another strong component.

**Theorem 4.** *Let $K$ be a proper cone and let $A \in \pi(K)$. In order that $A$ is $K$-irreducible it is necessary and sufficient that the following conditions are both satisfied*:

(a) *For any final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$, the join of all extreme rays which form the vertices of $\mathcal{C}$ is $K$.*

(b) *For any $x \in \text{Ext } K$, if the vertex $\Phi(x)$ has no access to a final strong component of $(\mathcal{E}, \mathcal{P})$, then the cone generated by all vertices of $(\mathcal{E}, \mathcal{P})$ which have access form $\Phi(x)$ intersects* int $K$.

**Theorem 5.** *Let $K$ be a polyhedral cone and let $A \in \pi(K)$. In order that $A$ is $K$-primitive, it is necessary and sufficient that for any final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P}(A, K))$, either $\mathcal{C}$ is a primitive digraph and the join of all extreme rays which form the vertices of $\mathcal{C}$ is $K$, or $\mathcal{C}$ is cyclically $m$-partite with ordered partition $\mathcal{E}_1, \ldots, \mathcal{E}_m$ for $V(\mathcal{C})$ for some $m > 1$ such that the join of all extreme rays in some (or, each) $\mathcal{E}_j$, $1 \leq j \leq m$, is $K$.*

By Theorems 4 and 5 we can say that the $K$-irreducibility or $K$-primitivity of $A$ is completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ together with a knowledge of when a given finite collection $\{E_1, \ldots, E_j\}$ of extreme rays satisfies $E_1 \vee \cdots \vee E_j = K$.

It is useful to introduce the concept of local exponent. For any $A \in \pi(K)$, not necessarily $K$-primitive or $K$-irreducible, and any $0 \neq x \in K$, by the *local exponent of $A$ at $x$*, denoted by $\gamma(A, x)$, we mean the smallest positive integer $k$ such that $A^k x \in \text{int } K$. If no such $k$ exists, we set $\gamma(A, x)$ equal $\infty$. Clearly, $A$ is $K$-primitive if and only if the set of local exponents of $A$ is bounded above; in this case, $\gamma(A)$ is equal to the maximum local exponent. By the work of the early paper [Bar 1], we also know that the $K$-primitivity of $A$ is equivalent to the apparently weaker condition (which is also the original definition adopted by Barker for $K$-primitivity) that all local exponents of $A$ are finite.

**Theorem 6.** *Let $A \in \pi(K)$ and let $0 \neq x \in K$. Denote by $\mathcal{E}(\Phi(x))$ the set of all extreme rays that lie in $\Phi(x)$. For any $\mathcal{S} \subseteq \mathcal{E}$, let $\Delta(\mathcal{S})$ denote the set $\{E \in \mathcal{E} : (G, E)$ is a $\mathcal{P}$-arc for some $G \in \mathcal{S}\}$ and write $\Delta^k(S)$ for $\Delta(\Delta^{k-1}(S))$, $k = 2, 3, \ldots$. Then*

*$\gamma(A,x)$ is finite if and only if there is a positive integer $k$ such that the join of all extreme rays that belong to $\Delta^k(\mathcal{E}(\Phi(x))$ equals $K$. If such $k$ exists, then the least possible value of $k$ is equal to $\gamma(A,x)$.*

**Theorem 7.** *Let $K$ be a polyhedral cone, and let $A$ be $K$-primitive. Suppose that the digraph $(\mathcal{E},\mathcal{P}(A))$ is strongly connected and cyclically $p$-partite with ordered partition $\mathcal{E}_1,\ldots,\mathcal{E}_p$ for $\mathcal{E}$ for some $p \geq 1$. For $j = 1,\ldots,p$, let $m_j, n_j$ denote respectively the cardinality of $\mathcal{E}_j$ and the dimension of the polyhedral cone generated by the vectors that belong to the extreme rays in $\mathcal{E}_j$. Also let $s$ denote the length of the shortest circuit in $(\mathcal{E},\mathcal{P}(A))$. Then $\gamma(A) \leq \max_{1\leq j\leq p}(pm_j + s(n_j-2))$.*

In Theorems 4, 5 and 6, besides given the digraph $(\mathcal{E},\mathcal{P}(A))$, we also assume that it is possible to determine whether any given finite collection of extreme vectors $x_1,\ldots,x_p$ satisfy $x_1+\cdots+x_p \in \text{int } K$. Certainly, if $K$ is given up to combinatorial equivalence, then the latter can be determined. In fact, we have the following:

**Theorem 8.** *Let $K_1$, $K_2$ be proper cones each with a bijective duality operator. Then $K_1$ and $K_2$ are combinatorial equivalent if and only if there exists a bijection $\varphi : \mathcal{E}(K_1) \longrightarrow \mathcal{E}(K_2)$ such that for any positive integer $p$ and any $E_1,\ldots,E_p \in \mathcal{E}(K_1)$, $E_1 \vee \cdots \vee E_p = K_1$ if and only if $\varphi(E_1) \vee \cdots \vee \varphi(E_p) = K_2$.*

## 2. PROOFS

**Proof of Theorem 1.** (b) $\Longrightarrow$ (c): Consider any $x \in \text{Ext } K$. If $Ax = 0$, then clearly $\Phi(Ax) = \{0\} \subseteq \Phi(Bx)$. Otherwise, we can write $Ax = y_1 + \cdots + y_r$, where $y_1,\cdots,y_r \in \text{Ext } K$. Then, for $j = 1,\ldots,r$, $(\Phi(x),\Phi(y_j))$ are $\mathcal{P}(A)$-arcs, and by condition (b) they are also $\mathcal{P}(B)$-arcs, i.e., $y_1,\ldots,y_r \in \Phi(Bx)$. So we have $Ax \in \Phi(Bx)$, or equivalently, $\Phi(Ax) \subseteq \Phi(Bx)$.

(c) $\Longrightarrow$ (d): Consider any $0 \neq x \in K$. We can write $x = x_1 + \cdots + x_s$, where $x_1,\ldots,x_s \in \text{Ext } K$. By condition (c), for each $i = 1,\ldots,s$, there exists $\alpha_i > 0$ such that $Ax_i \overset{K}{\leq} \alpha_i Bx_i$, where $\overset{K}{\leq}$ denotes the partial ordering of $\mathbb{R}^n$ induced by $K$. Let $\alpha = \max\{\alpha_1,\ldots,\alpha_s\}$. Then $\alpha > 0$ and we have $Ax \overset{K}{\leq} \alpha Bx$, so $\Phi(Ax) \subseteq \Phi(Bx)$.

(d) $\Longrightarrow$ (e): Suppose $(F,G)$ is a $\mathcal{P}(A)$-arc. Take any $x$ from $\text{ri } F$, the relative interior of $F$. Then $F = \Phi(x)$ and we have

$$G \subseteq \Phi(AF) = \Phi(Ax) \subseteq \Phi(Bx) = \Phi(BF).$$

where the inclusion holds by condition (c). So $(F, G)$ is also a $\mathcal{P}(B)$-arc.

(e) $\Longrightarrow$ (b): Obvious.

This establishes the equivalence of conditions (b)–(e).

The implications (a) $\Longrightarrow$ (d) $\Longrightarrow$ (f) are known and are not difficult to show (see [Tam 3]). ■

**Proof of Theorem 2**. If we are given the digraph $(\mathcal{F}', \mathcal{P}(A, K))$ and $(\mathcal{F}', \mathcal{P}(I, K))$, we can determine the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ as follows: The set $\mathcal{E}$ consists of precisely all those elements $E \in \mathcal{F}'$ for which there is no $F \in \mathcal{F}'$, $F \neq E$, such that $(E, F)$ is an arc of $(\mathcal{F}', \mathcal{P}(I, K))$. For any $E_1, E_2 \in \mathcal{E}$, $(E_1, E_2)$ is an arc in $(\mathcal{E}, \mathcal{P}(A, K))$ if and only if $(E_1, E_2)$ is an arc in $(\mathcal{F}', \mathcal{P}(A, K))$. Conversely, suppose we are given the digraph $(\mathcal{E}, \mathcal{P}(A, K))$. Consider any $F \in \mathcal{F}'$. We want to determine all those faces $G \in \mathcal{F}'$ for which $(F, G)$ is a $\mathcal{P}(A, K)$-arc, or equivalently, $\Phi(AF) \supseteq G$. If $x \in \text{Ext } F$, then from the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, we can determine all the extreme vectors of $\Phi(Ax)$. By elementary properties of faces (and of a cone-preserving map), one can readily show that $\Phi(AF)$ is equal to $\Phi(S)$, where $S = \bigcup \text{Ext } \Phi(Ax)$, where the union is taken over all $x \in \text{Ext } F$. (Note, however, that $\Phi(AF)$ may contain extreme vectors that are not in $S$.) The vertex $\Phi(AF)$ of $(\mathcal{F}', \mathcal{P}(I, K))$ can be captured by the following property: for any $y \in S$, $(\Phi(AF), \Phi(y))$ is an arc of $(\mathcal{F}', \mathcal{P}(I, K))$, and there does not exist $H \in \mathcal{F}'$ with the same property such that $(\Phi(AF), H)$ is an arc of $(\mathcal{F}', \mathcal{P}(I, K))$. Then for any $G \in \mathcal{F}'$, $(F, G)$ is a $\mathcal{P}(A, K)$-arc if and only if $(\Phi(AF), G)$ is an arc of $(\mathcal{F}', \mathcal{P}(I, K))$. So from the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ (and $(\mathcal{F}', \mathcal{P}(I, K))$) we can obtain the digraph $(\mathcal{F}', \mathcal{P}(A, K))$. Similarly, from $(\mathcal{E}, \mathcal{P}(A, K))$ we can also determine the digraphs $(\mathcal{E}, \mathcal{I}(A, K))$ and $(\mathcal{F}', \mathcal{I}(A, K))$. ■

**Proof of Theorem 3**. (i) First, we contend that for any positive integer $j$, we have $A^j \in \text{cl}_{\pi(K)}(\Phi(B^j))$. We proceed by induction on $j$. For $j = 1$, this holds by our assumption. Consider any $j \geq 2$ and suppose that we already have $A^{j-1} \in \text{cl}_{\pi(K)}(\Phi(B^{j-1}))$. By [Tam 3, Theorem 3.2], the latter condition implies that we have $A^{j-1}x \in \text{cl}_K(\Phi(B^{j-1}x))$ for all $x \in K$. Then, by [Tam 3, Theorem 3.3(b) and Theorem 3.2], we have $A^j x \in \text{cl}_K(\Phi(AB^{j-1}x)) \subseteq \text{cl}_K(\Phi(B^j x))$ for all $x \in K$, and by [Tam 3, Theorem 3.2] again, we have $A^j \in \text{cl}_{\pi(K)}(\Phi(B))$. This proves our contention.

Now let $\gamma$ be the exponent of $A$. Then $A^\gamma$ is $K$-positive. But by our contention we have $A^\gamma \in \text{cl}_{\pi(K)}(B^\gamma)$, so necessarily $B^\gamma$ is also $K$-positive. This proves that $B$ is $K$-primitive and $\gamma(B) \leq \gamma(A)$.

Part (ii) clearly follows from part(i). ■

[Before coming to the proofs of Theorems 4 and 5, I would like to point out that now I have some reservations on the validity of these results for a proper cone $K$. (If $K$ is polyhedral, I have no doubt that the results are correct.) In the proofs, we assume that if $\Phi(x)$ is any vertex in $(\mathcal{E}, \mathcal{P})$, then there is a path from $\Phi(x)$ to some vertex that belongs to a final strong component. If $(\mathcal{E}, \mathcal{P})$ is a finite graph, which is the case if $K$ is polyhedral, certainly there is no problem. But for a general proper cone $K$, $(\mathcal{E}, \mathcal{P})$ may have infinitely many extreme rays. Maybe the assertion is still correct, because $K$ is of finite dimension. But it requires a proof. Next, it is well-known that every finite strongly connected digraph is either primitive or cyclically $m$-partite for some integer $m \geq 2$. I am not sure whether the corresponding result still holds for an infinite strongly connected digraph, or in particular for a strong component of $(\mathcal{E}, \mathcal{P})$. In (the statement and the proof of) Theorem 5 I have already assumed that the answer to the latter question is in the affirmative. Also, in parts of the proofs, I need to consider the cone $C$ generated by extreme vectors that belong to the vertices of some final strong component. It is true that the cone $C$ is invariant under $A$. But I am not sure whether $C$ has to be closed.]

**Proof of Theorem 4.** "Only if": Suppose that there exists a final strong component $\mathcal{C}$ such that the join of all extreme rays which form the vertices of $\mathcal{C}$ is not equal to $K$. Let $C$ denote the cone generated by the extreme rays that form the vertices of $\mathcal{C}$. Consider any $x \in \text{Ext } C$. Since $A$ is $K$-irreducible, $Ax$ is a nonzero vector of $K$ and so it can be written as a positive linear combination of certain extreme vectors of $K$. Notice that if one of the extreme vectors which appears in this representation lies outside $C$, then in the digraph $(\mathcal{E}, \mathcal{P}(A))$ there must exist an arc from $\Phi(x)$ to some vertex not belonging to $\mathcal{C}$, which contradicts the assumption that $\mathcal{C}$ is a final strong component. Hence, each extreme vectors that appear in the representation of $Ax$ belongs to $C$; so $Ax$ itself also belongs to $C$. Since this is true for each extreme vector $x$ of $C$, we must have $AC \subseteq C$.

Since the join of all vertices of $\mathcal{C}$ is not equal to $K$, we have $C \subseteq \partial K$. But $C$ is invariant under $A$, so $\Phi(C)$ is a nontrivial $A$-invariant face of $K$. This contradicts the assumption that $A$ is $K$-irreducible.

"If" part: To prove that $A$ is $K$-irreducible, it suffices to show that $I + A$ is $K$-primitive, or equivalently, to show that for any $x \in \text{Ext } K$, there exists a positive integer $m$ such that $(I + A)^m x \in \text{int } K$.

Consider any $\Phi(x) \in \mathcal{E}$. Clearly there exists a path in $(\mathcal{E}, \mathcal{P})$ from $\Phi(x)$ to some vertex $\Phi(y)$ that belongs to a final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$. Hence, there exists a positive integer $p$ such that $y \in \Phi(A^p x)$. Let $C$ denote the cone generated by the extreme rays that form the vertices of $\mathcal{C}$. As done in the proof of the "only if" part, we have $AC \subseteq C$. Indeed, $A|_{\text{span } C}$ is irreducible with respect to $C$, as $(\mathcal{E}(C), \mathcal{P}(A|_{\text{span } C}, C))$ equals $\mathcal{C}$ and $\mathcal{C}$ is strongly connected. So, there exists a positive integer $q$ such that $(I + A)^q y \in \text{ri } C \subseteq \text{int } K$, where the last inclusion holds by our assumption on the strong components of $(\mathcal{E}, \mathcal{P})$. Thus, we have $(I + A)^{p+q} x \in \text{int } K$, as desired. This completes the proof. ■

By abuse of language, for any $\mathcal{T} \subseteq \mathcal{E}(K)$, we shall use $\Phi(\mathcal{T})$ to denote $\Phi(\bigcup E)$, where the union is taken over all extreme rays $E \in \mathcal{T}$.

The proofs of Theorem 5 and 6 depend on the following:

**Lemma 1.** *Let $K$ be a proper cone, and let $A \in \pi(K)$. If $F \in \mathcal{F}'$ and $\mathcal{T} \subseteq \mathcal{E}$ are such that $F = \Phi(\mathcal{T})$, then $\Phi(AF) = \Phi(\Delta(\mathcal{T}))$, where $\Delta(\mathcal{T})$ has the same meaning as defined in Theorem 6.*

*Proof.* For any $E \in \Delta(\mathcal{T})$, by definition, there exists $E' \in \mathcal{T}$ such that $(E', E)$ is a $\mathcal{P}(A)$-arc, i.e., $E \subseteq \Phi(AE')$. But $\Phi(\mathcal{T}) = F$, so we have $E \subseteq \Phi(AF)$. This establishes the inclusion $\Delta(\mathcal{T}) \subseteq \mathcal{E}(\Phi(AF))$, and hence also $\Phi(\Delta(\mathcal{T})) \subseteq \Phi(AF)$.

To prove the reverse inclusion, let $E_1, \ldots, E_k \in \mathcal{T}$ be such that $F = E_1 \vee \cdots \vee E_k$. Then

$$\begin{aligned}
\Phi(AF) &= \textstyle\bigvee_{i=1}^{k} \Phi(AE_i) \\
&= \textstyle\bigvee_{i=1}^{k} \Phi(\Delta(\{E_i\})) \\
&= \Phi(\Delta(\{E_1, \ldots, E_k\})) \\
&\subseteq \Phi(\Delta(\mathcal{T})),
\end{aligned}$$

where the second equality holds as it is clear that for any $E \in \mathcal{E}$, $\Delta(E) = \mathcal{E}(\Phi(AE))$. This completes the proof. ■

**Proof of Theorem 5.** "If" part: To prove that $A$ is $K$-primitive, it suffices to show that for any $x \in \text{Ext } K$ there exists a positive integer $l$ such that $A^l x \in \text{int } K$. Consider any $x \in \text{Ext } K$. As shown in the proof for the "if" part of Theorem 4, there exists a positive integer $p$ such that $\Phi(A^p x) \supseteq \Phi(y)$ for some extreme ray $\Phi(y)$ which is a vertex of some final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$. As in the proof of Theorem 4 we use $C$ to denote the cone generated by the extreme vectors that belong to extreme rays in $\mathcal{C}$. As already done in the proof of Theorem 4, $A|_{\text{span } C}$ must be $C$-irreducible. If the digraph $\mathcal{C}$ is primitive, then clearly $A|_{\text{span } C}$ is $C$-primitive, and so there exists a positive integer $q$ such that $A^q y \in \text{ri } C \subsetneq \text{int } K$, where the last inclusion follows from our assumption on the final strong components of $(\mathcal{E}, \mathcal{P})$. But then we have $\Phi(A^{p+q}x) \supseteq \Phi(A^q y) = K$, i.e., $A^{p+q}x \in \text{int } K$, as desired. So it remains to consider the case when $\mathcal{C}$ is cyclically $m$-partite with $m > 1$. By our hypothesis, there is an ordered partition $\mathcal{E}_1, \ldots, \mathcal{E}_m$ for $\mathcal{E}(\mathcal{C})$ such that for some $j$, $1 \leq j \leq m$, the join of all extreme rays in $\mathcal{E}_j$ is $K$. Since $\mathcal{C}$ is strongly connected, there is a path in $(\mathcal{E}, \mathcal{P})$ from $\Phi(y)$ to $\Phi(u)$ for some vertex $\Phi(u) \in \mathcal{E}_j$. Hence, there exists a positive integer $q$ such that $\Phi(A^q y) \supseteq \Phi(u)$. From the known theory of strongly connected digraph (see [B–R, ?]), there exists $r_0 \in \mathbb{Z}_+$ such that for all $r \in \mathbb{Z}_+$, $r \geq r_0$, there exists a directed walk of length $rm$ in $\mathcal{C}$ from $\Phi(u)$ to any vertex of $\mathcal{E}_j$. But the join of all extreme rays in $\mathcal{E}_j$ is $K$, so this means that $\Phi(A^{r_0 m}u) = K$. Thus, we have $A^{p+q+r_0 m}x \in \text{int } K$, as desired.

"Only if" part: Assume to the contrary that there exists a final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$ which is $m$-cyclic and with the ordered partition $\mathcal{E}_1, \ldots, \mathcal{E}_m$ for $\mathcal{E}(\mathcal{C})$ such that the join of all extreme rays in some $\mathcal{E}_j$ is not equal to $K$. (If $\mathcal{C}$ is primitive, then $m = 1$ and our argument still covers this case.) Take any $\Phi(x)$ from $\mathcal{E}_j$. Since $\mathcal{C}$ is a final strong component and is $m$-cyclic (and for any positive integer $k$ and any $\mathcal{S} \subseteq \mathcal{E}$, $\Delta^k(\mathcal{S})$ consists of precisely all those $E \in \mathcal{E}$ for which there is a directed walk of length $k$ from some vertex in $\mathcal{S}$ to $E$), clearly we have $\Delta^{rm}(\mathcal{E}(\Phi(x)) \subseteq \mathcal{E}_j$ for all positive integers $r$. By Lemma 1, this implies that $A^{rm}x \in \Phi(\mathcal{E}_j) \subseteq \partial K$ for all positive integers $r$, where the inclusion follows from our assumption on $\mathcal{E}_j$. This contradicts the $K$-primitivity of $A$. ■

**Proof of Theorem 6.** Consider any $0 \neq x \in K$. Applying Lemma 1 with $F = \Phi(x)$ and $\mathcal{T} = \mathcal{E}(\Phi(x))$, we obtain $\Phi(Ax) = \Phi(\Delta(\mathcal{E}(\Phi(x))))$. Proceeding inductively and using Lemma 1 repeatedly, we can show that $\Phi(A^k x) = \Phi(\Delta^k(\mathcal{E}(\Phi(x))))$ for all positive integers $k$. Thus, $A^k x \in \text{int } K$ if and only if $\Phi(\Delta^k(\mathcal{E}(\Phi(x)))) = K$, or if and only if the join of all extreme rays in $\Delta^k(\mathcal{E}(\Phi(x)))$ equals $K$. Hence our theorem follows. ■

The proof of Theorem 7 depends on the following:

**Lemma 2.** *Let $K$ be a proper cone in $\mathbb{R}^n$ $(n \geq 2)$, and let $A \in \pi(K)$. Assume that the digraph $(\mathcal{E}, \mathcal{P}(A))$ is strongly connected. If there is a positive integer $l$ such that $A^l$ is $K$-irreducible and there is a closed directed walk in $(\mathcal{E}, \mathcal{P})$ of length $l$, then*

*A is K-primitive. If, in addition, K is a polyhedral cone with m extreme rays and s is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A))$, then $\gamma(A) \leq m + s(n-2)$.*

*Proof.* Suppose that $\Phi(x_1), \ldots, \Phi(x_q)$ form the distinct vertices of a closed directed walk of length $l$ in $(\mathcal{E}, \mathcal{P})$. (Note that $q \leq l$, where the equality need not hold.)

Claim: For any $j$, $1 \leq j \leq q$, $A^{(n-1)l}x_j \in \text{int } K$.

*Proof of Claim*: Without loss of generality, consider $j = 1$. Since there is a (closed) directed walk of length $l$ from $\Phi(x_1)$ to itself, we have $\Phi(A^l x_1) \supseteq \Phi(x_1)$. Applying the powers of $A^l$ successively to both sides, we obtain an increasing chain of faces:

$$\Phi(x_1) \subseteq \Phi(A^l x_1) \subseteq \Phi(A^{2l}x_1) \subseteq \cdots.$$

For any positive integer $k$ for which $\Phi(A^{(k-1)l}x) \neq K$, by the $K$-irreducibility of $A^l$, we have the strict inclusion $\Phi(A^{(k-1)l}x_1) \subset \Phi(A^{kl}x_1)$ and hence $\dim \Phi(A^{kl}x_1) \geq \dim \Phi(A^{(k-1)l}) + 1$. But, to begin with, $\dim \Phi(x_1) = 1$, so we must have $\dim \Phi(A^{(n-1)l}x_1) = n$, i.e., $A^{(n-1)l}x_1 \in \text{int}\, K$.

Next, we show that for any $x \in \text{Ext}\, K$, there is a positive integer $p$ (depending on $x$) such that $A^p x \in \text{int}\, K$. Consider any $x \in \text{Ext } K$. Since $(\mathcal{E}, \mathcal{P})$ is strongly connected, there is a path, say of length $w$, from $\Phi(x)$ to one of the vertices $\Phi(x_1), \ldots, \Phi(x_q)$; say, $\Phi(x_i)$. Then we have $\Phi(A^w x) \supseteq \Phi(x_i)$ and hence $A^{w+(n-1)l}x \in \text{int } K$. Since each nonzero vector of $K$ can be written as a sum of extreme vectors, from the above, it follows that for any $0 \neq x \in K$, there is a positive integer $p$ such that $A^p x \in \text{int}\, K$, i.e., all local exponents of $A$ are finite. By [Bar 1] it follows that $A$ is $K$-primitive.

*Last Part.* Since $A$ is $K$-primitive, all positive powers of $A$, and in particular $A^s$, must be $K$-irreducible. Now repeat the above argument by replacing the closed directed walk of length $l$ by a circuit of shortest length. Then for any vertex $\Phi(y)$ that lies in the shortest circuit, we have $A^{(n-1)s}y \in \text{int } K$ and hence also $A^{m+(n-2)s}y \in \text{int}\, K$, as $m + (n-2)s \geq (n-1)s$. Moreover, for any vertex $\Phi(x)$ that lies outside the shortest circuit, there is a path of length $w \leq m - s$ from $\Phi(x)$ to one of the vertices that belong to the circuit. So we have

$$A^{m+(n-2)s}x = A^{m-s-w}(A^{(n-1)s}(A^w x)) \in \text{int } K.$$

This shows that $\gamma(A) \leq m + s(n-2)$. ■

**Proof of Theorem 7**. For each $j = 1, \ldots, p$, let $C_j$ denote the polyhedral cone generated by the vectors that belong to the extreme rays in $\mathcal{E}_j$. Clearly, if

$\Phi(y) \in \mathcal{E}_j$, then $Ay$ can be written as a positive linear combination of extreme vectors of $K$ that belong to the extreme rays in $\Delta(\Phi(y))$ and hence in $\mathcal{E}_{j+1}$ (where $\mathcal{E}_{p+1}$ is taken to be $\mathcal{E}_1$); that is, $Ay \in C_{j+1}$. Since this is true for each extreme vector $y$ of $C_j$, we have $AC_j \subseteq C_{j+1}$, and hence $A^p|_{\text{span}\, C_j} \in \pi(C_j)$ for $j = 1, \ldots, p$. Note that the digraph $(\mathcal{E}(C_j), \mathcal{P}(A^p|_{\text{span}\ C_j}))$ has vertex set $\mathcal{E}_j$, and for any $E_1, E_2 \in \mathcal{E}_j$, $(E_1, E_2)$ is an arc if there is a path in $(\mathcal{E}, \mathcal{P}(A, K))$ of length $p$ from $E_1$ to $E_2$. As $\mathcal{E}_j$ is one of the sets in the ordered partition of $\mathcal{E}$, from the theory of strong connected digraphs, we know the digraph $(\mathcal{E}(C_j), \mathcal{P}(A^p|_{\text{span}\ C_j}))$ is primitive. So each $A^p|_{\text{span}\ C_j}$ is $C_j$-primitive. Now for each $j$, applying Lemma 2 to $A^p|_{\text{span}\ C_j}$ and noting that the length of the shortest circuit in $(\mathcal{E}(C_j), \mathcal{P}(A^p|_{\text{span}\ C_j}))$ is at most $s/p$, we obtain $\gamma(A^p|_{\text{span}\ C_j}) \leq m_j + (s/p)(n_j - 2)$. Hence, for each $j = 1, \ldots, p$, we have $(A^p)^{m_j + (s/p)(n_j - 2)}x = A^{pm_j + s(n_j - 2)}x \in \text{ri}\ C_j$ whenever $\Phi(x) \in \mathcal{E}_j$. Since $A$ is $K$-primitive, by Theorem 5, we have ri $C_j \subseteq$ int $K$ for each $j$. So the local exponent of $A$ at each extreme vector of $K$ cannot exceed $\max_{1 \leq j \leq p}(pm_j + s(n_j - 2))$, hence we have $\gamma(A) \leq \max_{1 \leq j \leq p}(pm_j + s(n_j - 2))$. ■

**Proof of Theorem 8.** "Only if" part: Obvious.

"If" part: Observe that if $M$ is a subset of $\mathcal{E}(K_1)$ maximal with respect to the property that $\bigvee\{E : E \in M\} \neq K$, then the positive hull of the extreme rays in $M$ is a maximal face of $K_1$. Indeed, every maximal face of $K_1$ can be obtained in this way. The same can also be said for $K_2$. The given bijection $\varphi$ between $\mathcal{E}(K_1)$ and $\mathcal{E}(K_2)$ clearly induces a bijection between the maximal faces of $K_1$ and those of $K_2$. Now since the duality operator $d_{K_1}$ (also $d_{K_2}$) is bijective, each face of $K_1$ (also, of $K_2$) is an intersection of maximal faces. (For a proof, use [Tam 3, Lemma 5.12].) In other words, each face of $K_1$ (respectively, of $K_2$) is the positive hull of the extreme rays belonging to an intersection of subsets of $\mathcal{E}(K_1)$ (respectively, of $\mathcal{E}(K_2)$), each maximal with respect to the property that the join of its extreme rays is not equal to the whole cone. Hence, $\varphi$ also induces a bijection between the faces of $K_1$ and those of $K_2$. And, moreover, it is easy to see that the latter bijection also preserves the inclusion relation. Therefore, the cones $K_1$ and $K_2$ are combinatorially equivalent. ■

## 3. EXAMPLES, REMARKS AND OPEN QUESTIONS

The following example shows that if $A \in \pi(K)$, then the $K$-irreducibility or $K$-primitivity of $A$ is not completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$; it also depends on $K$.

By a *minimal cone* we mean a polyhedral cone whose number of extreme rays equals the dimension of the cone plus 1.

**Example 1.** Let $K$ be a minimal cone in $\mathbb{R}^5$ generated by extreme vectors $x_1, \ldots, x_6$ that satisfy $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$. Let $A$ be the 5-by-5 matrix given by:

$$\begin{aligned} Ax_1 &= x_2 + x_3, \ Ax_2 = x_3 + x_1, \ Ax_3 = x_1 + x_2, \\ Ax_4 &= x_5 + x_6, \ \text{ and } \ Ax_5 = x_6 + x_4. \end{aligned}$$

Then $Ax_6 = A(x_1 + x_2 + x_3) - A(x_4 + x_5) = x_4 + x_5$. So $A \in \pi(K)$. By [Tam 1, Theorem 4.1], the maximal faces of $K$ are precisely the subcones generated by four extreme vectors of $K$, two from each of the subsets $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. So the nontrivial faces of $K$ are simplicial. In particular, $\Phi(Ax_1)$, which is $\Phi(x_2 + x_3)$, contains precisely the extreme vectors $x_2$, $x_3$. Hence, in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, there are arcs $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ but no other arcs with initial vertex $\Phi(x_1)$. In this manner, we can determine all the arcs of $(\mathcal{E}, \mathcal{P}(A, K))$. It turns out that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by the following diagram:

$$\begin{array}{ccccccc} & \Phi(x_1) & & \qquad & & \Phi(x_4) & \\ / & & \backslash & & / & & \backslash \\ \Phi(x_2) & \text{——} & \Phi(x_3) & & \Phi(x_5) & \text{——} & \Phi(x_6) \end{array} \ ,$$

where we use $\Phi(x_i) — \Phi(x_j)$ to denote a pair of arcs $(\Phi(x_i), \Phi(x_j))$ and $(\Phi(x_j), \Phi(x_i))$. One can also readily check that $A$ is $K$-primitive with $\gamma(A) = 2$. [This example also shows that, when $A$ is $K$-primitive, the undirected graph of $(\mathcal{E}, \mathcal{P}(A, K))$ need not be connected.]

On the other hand, it is clear that one can also find a 6-by-6 nonnegative matrix, whose digraph is the same as the above one. Since the digraph is not strongly connected, any such nonnegative matrix is not even irreducible, not to say, primitive.

In [B–T, the paragraph following Proposition 1], an example is provided to show that, in general, the $K$-primitivity of $A$ does not imply the strong connectedness of $(\mathcal{E}, \mathcal{P})$, indeed not even that of $(\mathcal{E}, \mathcal{I})$. Below we are going to borrow the said example (but rewriting it and putting it in a more general form):

**Example 2.** Let $K$ be a minimal proper cone in $\mathbb{R}^4$ generated by the distinct extreme vectors $x_1, \ldots, x_5$ that satisfy the relation

$$2(x_1 + x_2 + x_3) = 3(x_4 + x_5).$$

Let $A$ be the 4-by-4 matrix given by:

$$Ax_1 = (x_1 + x_2)/2,\ Ax_2 = (x_2 + x_3)/2,\ Ax_3 = (x_3 + x_1)/2,$$
$$\text{and} \quad Ax_4 = (x_4 + x_5)/2.$$

After a little calculation, we obtain $Ax_5 = (x_4 + x_5)/2$; so $A \in \pi(K)$. Indeed, it is easy to check that $A$ is $K$-primitive and $\gamma(A) = 2$. Note that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. In fact, the digraph $(\mathcal{E}, \mathcal{P}(A))$ has two strongly connected components with vertex sets $V_1 = \{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$ and $V_2 = \{\Phi(x_4), \Phi(x_5)\}$. The induced subdigraph on $V_1$ is composed of loops at each of the vertices together with the 3-circuit $(\Phi(x_1), \Phi(x_2))$, $(\Phi(x_2), \Phi(x_3))$, $(\Phi(x_3), \Phi(x_1))$. The induced subdigraph on $V_2$ is complete. There is no arcs from $V_1$ to $V_2$, but there are arcs from each vertex of $V_2$ to all vertices of $V_1$.

Note that the matrix $A$ is singular. If we take $B = A + \varepsilon I$, then for $\varepsilon > 0$ sufficiently small, $B$ is nonsingular and $K$-primitive. Furthermore, we have $(\mathcal{E}, \mathcal{P}(B)) = (\mathcal{E}, \mathcal{P}(A))$ and $\gamma(B) = \gamma(A)$.

Now let $C$ be the 4-by-4 matrix given by:

$$Cx_1 = (x_1 + x_2)/2,\ Cx_2 = (x_2 + x_3)/2,\ Cx_3 = (x_3 + x_1)/2,$$
$$\text{and} \quad Cx_4 = \tfrac{2}{3}(x_2 + x_3).$$

Then, after a little calculation, we have $Cx_5 = \frac{2}{3}x_1$. So $C \in \pi(K)$. In fact, it is ready to see that $C$ is $K$-primitive and $\gamma(C) = 3$. Also, the digraph $(\mathcal{E}, \mathcal{P}(C))$ has three strongly connected components with vertex sets $\{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$, $\{\Phi(x_4)\}$ and $\{\Phi(x_5)\}$ respectively. The subdigraph on $\{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$ are the same as before. The remaining $\mathcal{P}(C)$-arcs are $(\Phi(x_4), \Phi(x_2))$, $(\Phi(x_4), \Phi(x_5))$ and $(\Phi(x_5), \Phi(x_1))$.

It is clear that if $G$ is any digraph with $m$ vertices, then we can always find an $m$-by-$m$ nonnegative matrix whose digraph is $G$. On the other hand, if $K$ is a nonsimplicial polyhedral cone with $m$ extreme rays, we need not be able to find some $A \in \pi(K)$ such that $(\mathcal{E}, \mathcal{P}(A, K))$ is the prescribed digraph $G$. There are certain constraints that have to be met in order that $G$ is of the form $(\mathcal{E}, \mathcal{P}(A, K))$. For instance, if $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ are any vectors of $K$ that satisfy $\Phi(x_1 + \cdots + x_r) = \Phi(y_1 + \cdots + y_s)$, then for any $A \in \pi(K)$ we must have $\Phi(Ax_1 + \cdots + Ax_r) = \Phi(Ay_1 + \cdots + Ay_s)$. Rewriting this in terms of digraphs, we have the following:

**Remark 2.** Suppose the digraph $(\mathcal{F}', \mathcal{P}(I, K))$ is given. Then for any $A \in \pi(K)$, the digraph of $(\mathcal{E}, \mathcal{P}(A, K))$ necessarily satisfies the following condition:

For any $\phi(x_1), \ldots, \Phi(x_r), \Phi(y_1), \ldots, \Phi(y_s) \in \mathcal{E}$, if the smallest element $F$ of $\mathcal{F}'$ ["smallest" in the sense of inclusion, which can be determined from the digraph $(\mathcal{F}', \mathcal{P}(I, K))$] with the property that $(F, \Phi(x_i))$ is a $\mathcal{P}(I, K)$-arc for $i = 1, \ldots, r$ is the same as the smallest element $G$ of $\mathcal{F}'$ with the property that $(G, \Phi(y_i))$ is a $\mathcal{P}(I, K)$-arc for $i = 1, \ldots, s$, then the smallest element $\tilde{F}$ of $\mathcal{F}'$ with the property that $(F, \Phi(w))$ is a $\mathcal{P}(I, K)$ arc for all $\Phi(w) \in \mathcal{E}$ such that $(\Phi(x_i), \Phi(w))$ is a $\mathcal{P}(A, K)$ arc for some $i = 1, \ldots, r$ is the same as the smallest element $\tilde{G}$ with the corresponding property, but with $y_1, \ldots, y_s$ in place of $x_1, \ldots, x_r$.

**Question 1.** Given a polyhedral cone $K$, determine all digraphs $G$ which are of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some $A \in \pi(K)$.

Question 1 can be considered as an "allow" question with $K$ fixed. One may also ask a similar question for which $K$ is not fixed. But, as mentioned above, any (finite) digraph is the digraph associated with some nonnegative matrix. So we should exclude the simplicial cones.

**Question 2.** Determine all (finite) digraphs $G$ which are of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some non-simplicial (polyhedral) cone $K$ and some $A \in \pi(K)$ (also, $K$-irreducible or $K$-primitive $A$).

Certainly, there are digraphs which are not of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some non-simplicial polyhedral cone $K$ and some $A \in \pi(K)$, for instance, any digraphs with three or less vertices. Below is a less trivial example:

**Example 3.** Consider the digraph $G$ which consists of two 3-circuits with one vertex in common. If $G$ is of the form $(\mathcal{E}, \mathcal{P}(A, K))$, then the only possible dimensions for the non-simplicial cone $K$ is 3 or 4. We treat both cases together. To be specific, suppose $(\mathcal{E}, \mathcal{P}(A, K))$ is represented by the following diagram:

$$\begin{array}{ccccccccc} & & \Phi(x_4) & & & & \Phi(x_2) & & \\ & \swarrow & & \nwarrow & & \nearrow & & \searrow & \\ \Phi(x_5) & & \longrightarrow & & \Phi(x_1) & & \longleftarrow & & \Phi(x_3) \end{array}$$

Since there is only one arc with initial vertex $\Phi(x_4)$, namely, $(\Phi(x_4), \Phi(x_5))$, we must have $A\Phi(x_4) = \Phi(x_5)$. Similarly, we have $A\Phi(x_2) = \Phi(x_3)$, $A\Phi(x_5) = A\Phi(x_3) = \Phi(x_1)$ and also $A\Phi(x_1) = \Phi(x_4) \vee \Phi(x_2)$, which is a 2-dimensional simplicial face. Note that the condition $A\Phi(x_5) = A\Phi(x_3) = \Phi(x_1)$ implies that $A$ maps $x_5$ and $x_3$ both into the extreme ray $\Phi(x_1)$; hence $A$ must be singular. Certainly, $AK$ is the polyhedral cone generated by the images of the extreme vectors of $K$ under $A$. So from the above information on $A$, we have, $AK = \text{pos}\{x_1, x_3, x_5, \alpha_2 x_2 + \alpha_4 x_4\}$, assuming $Ax_1 = \alpha_2 x_2 + \alpha_4 x_4$, where $\alpha_2, \alpha_4 > 0$; hence, rank $A = \dim AK \geq 3$. This clearly disposes of the case when $\dim K = 3$ (because $A$ is singular). So $K$ must be a 4-dimensional minimal cone and we have $\alpha_2 x_2 + \alpha_4 x_4 \in \text{span}\{x_1, x_3, x_5\}$; say,

$$\alpha_2 x_2 + \alpha_4 x_4 = \alpha_1 x_1 + \alpha_3 x_3 + \alpha_5 x_5. \tag{3.1}$$

Then the latter is the unique (up to multiples) linear relation for the extreme vectors $x_1, \ldots, x_5$ of the minimal cone $K$. Applying $A$ to both side of (3.1), we obtain

$$\alpha_2 \lambda_2 x_3 + \alpha_4 \lambda_4 x_5 = \alpha_1 \alpha_2 x_2 + \alpha_1 \alpha_4 x_4 + (\alpha_3 \lambda_3 + \alpha_5 \lambda_5) x_1, \tag{3.2}$$

where $\lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5 > 0$. From (3.1) and (3.2), $(\alpha_3, \alpha_5)$ is a nonzero multiple of $(\alpha_2 \lambda_2, \alpha_4 \lambda_4)$; so $\alpha_3,\ \alpha_5$ must be both positive or both negative. If $\alpha_3,\ \alpha_5$ are both negative, then we would arrive at a contradiction — namely, $x_1$ is not an extreme vector of $K$ if $\alpha_1 > 0$, or $K$ is not pointed if $\alpha_1 \leq 0$. So they are both positive. Then $\alpha_3 \lambda_3 + \alpha_5 \lambda_5 > 0$, and again by comparing (3.1) and (3.2), we obtain $\alpha_1 < 0$. Then from (3.2), it follows that $x_1$ is a positive linear combination of $x_2,\ x_3,\ x_4$ and $x_5$, which contradicts the assumption that $x_1$ is an extreme vector.

The following is another fundamental question:

**Question 3.** Let $K_1$ and $K_2$ be proper cones which are combinatorially equivalent. Is it true that for any digraph $G$, if there exists $A_1 \in \pi(K_1)$ such that $(\mathcal{E}, \mathcal{P}(A_1, K_1)) = G$, then there always exists $A_2 \in \pi(K_2)$ such that $(\mathcal{E}, \mathcal{P}(A_2, K_2)) = G$ ?

For any two square nonnegative matrices $A$, $B$ of the same size, it is clear that $A$ and $B$ have the same (usual) digraph if and only if so do $A^T$ and $B^T$. One may suspect that the corresponding result also holds for cone-preserving maps. [Recall that for a proper cone $K$ and any matrix $A$, $A \in \pi(K)$ if and only if $A^T \in \pi(K^*)$.] The following example shows that this is not true. It also shows that in general $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ does not imply $\Phi(A) = \Phi(B)$.

**Example 4.** Let $K$ be a proper cone whose dual cone $K^*$ is not facially exposed. Choose a non-exposed face $\Phi(z)$ of $K^*$. Let $w \in K^*$ be such that $\Phi(w)$ equals $\mathrm{cl}_{K^*}(\Phi(z))$, the exposed face of $K^*$ generated by $z$. Choose any $x \in \mathrm{int}\, K$. Take $A = xz^T$ and $B = xw^T$. Clearly $A, B \in \pi(K)$. By our choices of $w$ and $z$, for any $y \in K$, we have $z^T y = 0$ if and only if $w^T y = 0$. Hence, we have $\Phi(Ay) = \Phi(By)$ for any $y \in K$. By Corollary 1 this means that $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$. If $B \in \Phi(A)$, then there exists $\alpha > 0$ such that $A - \alpha B = x(z - \alpha w)^T \in \pi(K)$. But the face $\Phi(w)$ properly includes $\Phi(z)$, so $w \notin \Phi(z)$ and hence $z - \alpha w \notin K^*$. Take $y \in K$ such that $(z - \alpha w)^T y < 0$. Then $(A - \alpha B)y$ is a negative multiple of $x$, which is a contradiction. So we must have $B \notin \Phi(A)$. Finally, take any $u \in \mathrm{int}\, K^*$. Then $B^T u = (x^T u)w \notin \Phi(z) = \Phi(A^T u)$. By Corollary 1 again, we have $(\mathcal{E}, \mathcal{P}(A^T, K^*)) \neq (\mathcal{E}, \mathcal{P}(B^T, K^*))$.

However, using [Tam 3, Corollary 5.9] (and the argument of the above example), one can readily show the following:

**Remark 3.** Let $K$ be a proper cone. In order that for any $A, B \in \pi(K)$, we have

$$(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K)) \ \text{ iff } \ (\mathcal{E}, \mathcal{P}(A^T, K^*)) = (\mathcal{E}, \mathcal{P}(B^T, K^*))$$

it is necessary and sufficient that the duality operator $d_K$ be bijective (which is the case if $K$ is polyhedral).

On the other hand, the conditions "every face of $\pi(K)$ can be expressed as an intersection of simple faces" and "$d_{\pi(K)}$ is injective", which appear in Theorem 1 and Corollary 1, are not yet well understood and there are still some open problems involving these conditions. We refer the interested reader to [Tam 3, Section 6].

We call two cones $K_1$, $K_2$ *linearly isomorphic* if there exists a nonsingular linear transformation $T$ from span $K_1$ to span $K_2$, which maps $K_1$ onto $K_2$. We call the cones $K_1$, $K_2$ *combinatorially equivalent*, if their face lattices $\mathfrak{F}(K_1)$ and $\mathfrak{F}(K_2)$ are isomorphic (as lattices), or equivalently, the digraphs $(\mathcal{F}', \mathcal{P}(I, K))$ and $(\mathcal{F}', \mathcal{P}(I, K_2))$ are equal (up to graph isomorphism). [I don't know whether it is true that if $K_1$, $K_2$ are combinatorially equivalent cones, then so are $\pi(K_1)$ and $\pi(K_2)$.]

It is clear that linearly isomorphic cones are combinatorially equivalent, but the converse is not true. Here is an example:

**Example 5.** Let $e_j$, $j = 1, 2, 3$, denote the standard unit vectors of $\mathbb{R}^3$. Let $K_1$ be the polyhedral cone in $\mathbb{R}^3$ generated by the extreme vectors $e_1$, $e_2$, $e_3$, $2e_1+e_2-e_3$ and $e_1 + 2e_2 - e_3$. We are going to show that there exists a vector $u \in \mathbb{R}^3$ such that the polyhedral cone $K_2$ generated by the extreme vectors $e_1$, $e_2$, $e_3$, $2e_1+e_2-e_3$ and $u$ is not linearly isomorphic with $K_1$. Clearly, any 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent. In particular, the cones $K_1$, $K_2$ are combinatorially equivalent. We want to construct $u$ in such a way that the extreme ray $\Phi(u)$ is neighborly to $\Phi(2e_1 + e_2 - e_3)$ and $\Phi(e_2)$. If $T$ is a linear isomorphism which maps $K_2$ onto $K_1$, then $T$ must carry extreme rays to extreme rays. Certainly, $T$ maps neighborly extreme rays to neighborly extreme rays. So, there are eight choices for the action of $T$ on the extreme rays of $K_2$: the images of $\Phi(e_2)$, $\Phi(e_3)$, $\Phi(e_1)$, $\Phi(2e_1 + e_2 - e_3)$ under $T$, in this order, can be $\Phi(e_2)$, $\Phi(e_3)$, $\Phi(e_1)$, $\Phi(2e_1 + e_2 - e_3)$, or $\Phi(2e_1 + e_2 - e_3)$, $\Phi(e_1)$, $\Phi(e_3)$, $\Phi(e_2)$, or $\Phi(e_3)$, $\Phi(e_1)$, $\Phi(2e_1 + e_2 - e_3)$, $\Phi(e_1 + 2e_2 - e_3)$, and so forth. It is not difficult to show that once the action of $T$ on $\Phi(e_2)$, $\Phi(e_3)$, $\Phi(e_1)$ and $\Phi(2e_1 + e_2 - e_3)$ are known, $T$ is uniquely determined up to multiples. We choose $u$ in such a way that, for each of the eight choices, $Tu$ does not lie on the remaining extreme ray of $K_1$; that is the one which is different from $T\Phi(e_2)$, $T\Phi(e_3)$, $T\Phi(e_1)$ and $T\Phi(2e_1+e_2-e_3)$; then $TK_2 \neq K_1$. Clearly, such $u$ exists. Therefore, the cones $K_1$ and $K_2$ are not linearly isomorphic.

Putting it in another way, Theorem A tells us that if $K$ is non-simplicial, then there always exists $A \in \pi(K)$ such that $A$ is $K$-irreducible and $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. One may wonder whether in this case there is also a $K$-primitive matrix $A$ such that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. The answer turns out to be "no".

A proper cone is said to be *strictly convex* if every boundary vector is extreme.

**Remark 4.** Let $K$ be a strictly convex cone, and let $A \in \pi(K)$. If $A$ is $K$-primitive, then $(\mathcal{E}, \mathcal{P}(A))$ is strongly connected.

To see this, consider any extreme ray $\Phi(x)$ of $K$. We want to show that there is a path in $(\mathcal{E}, \mathcal{P})$ from the vertex $\Phi(x)$ to any other vertex of $\mathcal{E}$. Let $p$ denote the least positive integer such that $A^p x \in \text{int } K$. Then $x, Ax, \ldots, A^{p-1}x$ all belong to $\partial K$, and since $K$ is strictly convex, all of them are, in fact, (nonzero) extreme vectors of $K$. So there is a path in $(\mathcal{E}, \mathcal{P})$ passing through the vertices $\Phi(x), \Phi(Ax), \ldots, \Phi(A^{p-1}x)$

(and in this order). Note that there is a $\mathcal{P}$-arc from $\Phi(A^{p-1}x)$ to any vertex of $\mathcal{E}$, as $A(A^{p-1}x) \in \text{int } K$. Hence, there is a path form $\Phi(x)$ to any vertex of $\mathcal{E}$. This proves the strong connectedness of $(\mathcal{E}, \mathcal{P})$.

**Question 4.** If $K$ is a non-simplicial polyhedral cone, then does there always exist a $K$-primitive matrix $A$ such that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected ?

In [Bar, Theorem 2], it is proved that if $K$ is a polyhedral cone with $m$ extreme rays, then for any $A \in \pi(K)$, an equivalent condition for $A$ to be $K$-primitive is that $A^j$ is $K$-irreducible for $j = 1, 2, \ldots, 2^m - 1$. When the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected, by Lemma 2, we can replace the latter condition simply by "$A^s$ is $K$-irreducible, where $s$ is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$".

**Remark 5.** Let $K$ be a proper polyhedral cone with $m$ extreme rays. Let $A \in \pi(K)$ and suppose that $(\mathcal{E}, \mathcal{P})$ is strongly connected. Then $A$ is $K$-primitive if and only if $A^{m^2-2m+2}$ is $K$-positive.

It suffices to consider the "only if" part. Note that if $s$, the length of the shortest circuit, equals $m$, then necessarily the digraph $(\mathcal{E}, \mathcal{P})$ is simply a circuit of length $m$. But then $A$ must map the set $\mathcal{E}$ onto itself, i.e., $A$ is an automorphism on $K$, which contradicts the $K$-primitivity of $A$. So we must have $s \leq m - 1$. But we also have $n \leq m$, where $n = \dim K$. By Lemma 2 we have

$$\gamma(A) \leq m + s(n-2) \leq m + (m-1)(m-2) = m^2 - 2m + 2.$$

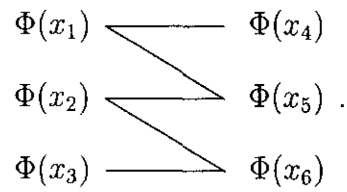
Therefore, $A^{m^2-2m+2}$ is $K$-positive.

The above remark can be found in [Niu, Corollary 4]. But there is a gap in the proof of its "only if" part. It is asserted that in this case $(\mathcal{E}, \mathcal{P})$ always has two circuits of different lengths (thus, $s \leq m - 1$), but this is not true. The problem at issue is related to the concept of a primitive digraph. By a *primitive digraph* we mean, as usual, a strongly connected digraph $G$ for which there exists a positive integer $k_0$ such that for all positive integers $k \geq k_0$, there is a directed walk of length $k$ in $G$ from $x$ to $y$ for any vertices $x$, $y$ of $G$. The smallest such positive integer $k_0$ is called the exponent of $G$ and is denoted by $\gamma(G)$. It is known that a strongly connected digraph is primitive if and only if the greatest common divisor of its circuit lengths is equal to 1. It is easy to show that if $(\mathcal{E}, \mathcal{P}(A))$ is primitive, then $A$ is $K$-primitive and $\gamma(A) \leq \gamma(\mathcal{E}, \mathcal{P}(A))$. So the natural question to ask is, whether it is true that if $A$ is $K$-primitive and $(\mathcal{E}, \mathcal{P}(A, K))$ is strongly connected, then $(\mathcal{E}, \mathcal{P}(A,K))$ is a primitive digraph. If the answer is "yes", then the gap in Niu's proof mentioned above can be removed, and moreover then we can readily recover Theorem D by the classical Dulmage and Mendelsohn theorem. The answer, however, turns out to be "no" as the following example shows.

**Example 6.** Let $K$ be a minimal proper cone in $\mathbb{R}^5$ generated by the extreme vectors $x_1, \ldots, x_6$ that satisfy $x_1+x_2+x_3 = x_4+x_5+x_6$. Let $A$ be the 5-by-5 matrix given by: $Ax_1 = 2x_4 + x_5$, $Ax_2 = x_5 + x_6$, $Ax_3 = x_6$, $Ax_4 = x_1$ and $Ax_5 = x_1 + x_2$. Then

$$Ax_6 = A(x_1 + x_2 + x_3) - Ax_4 - Ax_5 = x_2 + 2x_3.$$

So, clearly we have $A \in \pi(K)$. It turns out that $(\mathcal{E}, \mathcal{P}(A))$ is the following symmetric bipartite digraph with bipartition $\{\{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}, \{\Phi(x_4), \Phi(x_5), \Phi(x_6)\}\}$:




[In the above diagram, we use $\Phi(x_i)$ —— $\Phi(x_j)$ to represent a pair of arcs $(\Phi(x_i), \Phi(x_j))$ and $(\Phi(x_j), \Phi(x_i))$.] So, it is clear that the digraph $(\mathcal{E}, \mathcal{P}(A))$ is not primitive, and moreover it has only one circuit length, namely, 2. Using Theorem 6, one can determine the local exponent of $A$ at $x_1$ as follows:

$$\Delta(\Phi(x_1) = \{\Phi(x_4), \Phi(x_5)\} \text{ with } x_4 + x_5 \in \partial K,$$
$$\Delta^2(\Phi(x_1) = \{\Phi(x_1), \Phi(x_2)\} \text{ with } x_1 + x_2 \in \partial K,$$
$$\text{and} \quad \Delta^3(\Phi(x_1) = \{\Phi(x_4), \Phi(x_5), \Phi(x_6)\} \text{ with } x_4 + x_5 + x_6 \in \operatorname{int} K.$$

So we have $\gamma(A, x_1) = 3$. Similarly, one can show that

$$\gamma(A, x_2) = \gamma(A, x_5) = 2, \ \gamma(A, x_6) = 3, \ \text{ and } \ \gamma(A, x_3) = \gamma(A, x_4) = 4.$$

Hence, $A$ is $K$-primitive and $\gamma(A) = 4$.

Recall that a positive integer $\kappa$ is called the *critical exponent* of a normed space $(E, \|\cdot\|)$ (or of the norm on $E$) if the equalities $\|A^\kappa\| = \|A\| = 1$ imply that $\rho(A) = 1$, and if $\kappa$ is the smallest number with the indicated property. (Here $A$ stands for a linear operator on $E$ and $\|A\| = \sup_{0 \neq x \in E} \|Ax\|/\|x\|$.) In [B–L, p.67] an example of a norm on $\mathbb{R}^2$ is given such that with respect to this norm $\mathbb{R}^2$ has no critical exponent. Now we are going to show that one can construct an example of a proper cone $K$ in $\mathbb{R}^3$ for which $\gamma(K)$ is infinite.

**Example 7.** Let $\|\cdot\|$ denote the norm of $\mathbb{R}^2$ given in [B–L, p.67]. Let $K$ be the proper cone in $\mathbb{R}^3$ given by: $K = \{\alpha\binom{x}{1} : \alpha \geq 0 \text{ and } \|x\| \leq 1\}$. As show in [B–L], for each positive integer $k$, we can find some 2-by-2 real matrix $B_k$ such that $\|B_k\| = \|B_k^k\| = 1$ but $\|B_k^{k+1}\| < 1$. Let $A_k = B_k \oplus (1)$. Then it is easy to show that $A_k$ is $K$-primitive and $\gamma(A_k) = k$. Since $k$ can be arbitrarily large, this shows that for this $K$ we have $\gamma(K) = \infty$. It is also of interest to note that the $K$-primitive matrices $A_k$ obtained in this example are, in fact, all extreme matrices of the cone $\pi(K)$. The point is, each of them map infinitely many extreme rays of $K$ onto extreme rays.

Notice, however, that the proper cone $K$ considered in Example 7 is not strictly convex.

**Question 5.** Is it true that every strictly convex cone in $\mathbb{R}^n$ has finite exponent?

We do not even know the value of $\gamma(K_n)$ for the $n$-dimensional ice-cream cone $K_n$. Making use of the proof for the known fact that the critical exponent of an $n$-dimensional euclidean space is $n$ (see [B–L, Theorem 2.6.2]), one can readily show that $\gamma(G) \geq n$.

**Question 6.** Is the set $\{\gamma(K) : K \text{ is a polyhedral cone in } \mathbb{R}^n\}$ bounded above?

As far as we know, the following question posed by Barker [Bar 1] in 1972, now rephrased in terms of the concept of local exponent, is still open:

**Question 7.** Is it true that for any proper cone $K$ and any $K$-irreducible matrix $A$, the set $\{\gamma(A, x) : x \in K \text{ and } \gamma(A, x) \text{ is finite}\}$ is bounded above ?

Finally, we would like to point out that the tool of a minimal generating matrix, which is used by some people in the study of polyhedral cones (see, for instance, [B–F–H]), also has some connections with our study. Namely, if $K$ is a polyhedral cone and $A \in \pi(K)$, and if $B$ is the nonnegative matrix with the maximum number of positive entries that satisfies $AP = PB$, where $P$ is the minimal generating matrix for $K$ (i.e., its column vectors form a set of distinct representatives of the extreme rays of $K$), then the usual digraph of $B^T$ is equal to our digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$. Since the strong connectedness of $(\mathcal{E}, \mathcal{P}(A, K))$ implies the $K$-irreducibility of $A$ but not conversely, we recover the known relation between the $K$-irreducibility of $A$ and the irreducibility of $B$.

The following are some other ideas I have not yet pursued:

1. I guess primitive digraphs are precisely the digraphs $G$ which require the property that for every proper cone $K$ and every $A \in \pi(K)$, $A$ is $K$-primitive whenever $(\mathcal{E}(K), \mathcal{P}(A, K)) = G$.

2. In my long survey paper "A cone-theoretic approach to the spectral theory of positive linear operators ...", there appears in Theorem 6.3 a result which says that if $A \in \pi(K)$, where $K$ is a polyhedral cone with $m$ maximal faces, then there exists an $m$-by-$m$ nonnegative matrix $B$ and some $B$-invariant subspace $W$ of $\mathbb{R}^m$, $W \bigcap \text{int } \mathbb{R}^m_+ \neq \emptyset$, such that the cone-preserving maps $A \in \pi(K)$ and $B|_W \in \pi(W \bigcap \mathbb{R}^m_+)$ are equivalent. This result may also have some connection with our work, but I have not yet explored it.

3. We may also consider $A \in \pi(K_1, K_2)$, where $K_1$, $K_2$ are proper cones, possibly in different euclidean spaces. For each such $A$, we can associate with it two bipartite graphs: The first bipartite graph has bipartition $\{\mathcal{E}(K_1), \mathcal{E}(K_2)\}$ for which there is an edge joining $E_1$, $E_2$, where $E_1 \in \mathcal{E}(K_1)$ and $E_2 \in \mathcal{E}(K_2)$ if and only if $E_2 \subseteq \Phi(AE_1)$. Similarly, we can define a bipartite graph with bipartition $\{\mathcal{F}'(K_1), \mathcal{F}'(K_2)\}$. I think Theorem 1 and Corollary 1 also have corresponding results in this setting, because their proofs rely on [Tam 3], but the latter paper is done in this general setting. Of course, in this case, we do not have the concept of $K$-irreducibility or $K$-primitivity.

4. We may even work in the setting of non-linear cone-preserving maps, say, in the class of monotone homogeneous maps on a fixed proper cone, or even in a broader class. I think we can always associate $A$ with a digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$ defined in the same way as in this paper as long as $A$ is a map which preserves $K$ and possesses the following property: for all $x, y \in K$, if $\Phi(x) = \Phi(y)$ then $\Phi(Ax) = \Phi(Ay)$. (Is the latter property equivalent to, for all $x, y \in K$, if $\Phi(x) \subseteq \Phi(y)$, then $\Phi(Ax) \subseteq \Phi(Ay)$ ?), We can still have the concepts of $K$-irreducibility and $K$-primitivity and, I believe, many of our results can be carried over to this more general setting.

5. In the nonnegative matrix case, we can describe the $A$-invariant faces completely. (See my paper with Hans "On the invariant faces associated with a cone-preserving map.) If the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ for $A$ $(\in \pi(K))$ is given, I am wondering to what extent we can describe all the $A$-invariant faces ?

## References


[Bar 1] G.P. Barker, On matrices having an invariant cone, *Czechoslovak Math. J.* **22** (1972), 49–68.

[Bar 2] G.P. Barker, Theory of cones, *Linear Algebra Appl.* **39** (1981), 263–291.

[B–F–H] F. Burns, M. Fiedler and E. Haynsworth, Polyhedral cones and positive operators, *Linear Algebra Appl.* **8** (1974), 547–559.

[B–L] G.R. Belitskii and Yu.I. Lyubich, *Matrix Norms and Their Applications*; English transl.; Birkhäuser Verlag, Basel, 1988.

[B–R] R.A. Brualdi and H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991.

[B–T] G.P. Barker and B.S. Tam, Graphs for cone preserving maps, *Linear Algebra Appl.* **37** (1981), 199–204.

[Niu] S.Z. Niu, The index of primitivity of $K_m$-nonnegative operator, *Jour. of Beijing Univ. of Posts and Telecom.* **17** (1994), 95–98.

[D–M] A.L. Dulmage and N.S. Mendelsohn, Gaps in the exponent of primitive matrices, *Illinois J. Math.* **8** (1964), 642–656.

[S–V] H. Schneider and M. Vidyasagar, Cross-positive matrices, *SIAM J. Num. Anal.* **7** (1970), 508–519.

[Tam 1] B.S. Tam, Diagonals of convex cones, *Tamkang J. Math.* **14** (1983), 91–102.

[Tam 2] B.S. Tam, On the duality operator of a convex cone, *Linear Algebra Appl.* **64** (1985), 33–56.

[Tam 3] B.S. Tam, On the structure of the cone of positive operators, *Linear Algebra Appl.* **167** (1992), 65–85.

[T–B] B.S. Tam and G.P. Barker, Graphs and irreducible cone preserving maps, *Linear and Multilinear Algebra* **31** (1992), 19–25.

[Wie] H. Wielandt, Unzerlegbare, nicht negative Matrizen, *Math. Zeit.* **52** (1950), 642–645.

The following example was constructed in my initial attempt to determine $\gamma(K_n)$. Of course, it is superseded by the now known fact that $\gamma(K_n) \geq n$. I keep the example here in case it may be useful.

**Example.** Let $K_n$ be the $n$-dimensional ice-cream cone, $n \geq 3$, i.e., $K_n = \{(\xi_1, \ldots, \xi_n)^T : \xi_n \geq (\xi_1^2 + \cdots + \xi_{n-1}^2)^{1/2}\}$. Choose any two distinct extreme vectors $x_1$, $x_2$ of $K_n$. Suppose $z_2$ is the (unique) extreme vector of $K_n$ ($= K_n^*$) orthogonal to $x_2$. Let $z_1$ be any extreme vector of $K_n$, distinct from $z_2$ and not orthogonal to $z_1$. Let $A = x_1 z_1^T + x_2 z_2^T$. We are going to show that $A$ is $K$-primitive and $\gamma(A) = 3$. Let $y$ be the extreme vector of $K_n$ orthogonal to $z_1$. If $x$ is any (nonzero) extreme vector of $K_n$ which is not a multiple of $x_2$ or of $y$, then it is clear that $Ax \in \text{int } K$. In particular, $Ax_1 \in \text{int } K$. By calculation, we have $Ay_1 = (z_2^T y)x_2$, $Ax_2 = (z_1^T x_2)x_1$, where $z_2^T x_2$ are both positive numbers. So it is clear that $A$ is $K$-primitive and $\gamma(A) = 3$.

# 出席國際會議報告

會議名稱：第六屆數值域及數值半徑研討會

會議地點：美國奧本市奧本大學

會議時間：91 年 6 月 7 日--8 日

報 告 人：淡江大學數學系　譚必信

撰寫日期：91 年 6 月 26 日

《數值域及數值半徑研討會》每兩年舉辦一次，今年是第六屆，輪到在美國奧本大學舉行，主辦者爲該校的譚天祐教授及威廉瑪琍大學的李志光教授。

會議參加者約 35 人，都是來自世界各地的專家，比較特別的是，參加者當中有 10 位爲華人（源自香港大學的則有 5 位），而日本人也佔了 5 位。

在兩個全天的會議中，總共安排了 17 場半小時（含 5 分鐘發問時間）演講，都是在友好的氣氛下單場進行。感到興趣的演講不少，包括：T. Ando 的 "Convexoid matrices"、M.D. Choi 的 "Numerical ranges and dilation"、C.K. Li 的 "Linear operators on matrix algebra that preserve states, numerical range, and numerical radius"、Y.T. Poon 的 "Principal submatrices of a Hermitian matrix"、T.Y. Tam 的 "Star-shapedness of some generalized numerical ranges" 及 T. Yamazaki 的 "Properties of Aluthge transformations on operator norms”等等。本人獲益良多。

本人的演講是安排在會議的第二天下午，講題爲“The numerical range of a nonnegative matrix”。

這次研討會的參加者（包括本人）大部份都有參加接著於 6 月 10 日至 13 日在奧本大學舉行的第十屆國際線性代數學會會議。

攜回資料：會議議程及摘要一本。

# The Numerical Range of a Nonnegative Matrix

Chi-Kwong Li[1]
Department of Mathematics
College of William and Mary
Williamsburg, VA 23187
U.S.A.

Bit-Shun Tam[2]
Department of Mathematics
Tamkang University
Tamsui, Taiwan 25137
R.O.C.

Pei Yuan Wu[3]
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan 30050
R.O.C.

October 30, 2001

**Abstract**

We offer an almost self-contained development of Perron-Frobenius type results for the numerical range of an (irreducible) nonnegative matrix, rederiving and completing the previous work of Issos, Nylen and Tam, and Tam and Yang on this topic. We solve the open problem of characterizing nonnegative matrices whose numerical ranges are regular convex polygons with center at the origin. Some related results are obtained and some open problems are also posed.

2000 Mathematics Subject Classification: 15A60, 15A48.
Keywords: Numerical range; Nonnegative matrix; Irreducible real part; Numerical radius; Perron-Frobenius theory; Regular polygon; Sharp point.

[1]Research partially supported by a USA NSF grant.
[2,3]Supported by the National Science Council of the Republic of China

## 1 Introduction

By the classical Perron-Frobenius theory, if $A$ is a (square, entrywise) nonnegative matrix, then its spectral radius $\rho(A)$ is an eigenvalue of $A$ and there is a corresponding nonnegative eigenvector. If, in addition, $A$ is irreducible, then $\rho(A)$ is a simple eigenvalue and the corresponding eigenvector can be chosen to be positive. Moreover, for an irreducible nonnegative matrix with index of imprimitivity $m > 1$ (i.e., one having exactly $m$ eigenvalues with modulus $\rho(A)$), Frobenius has also obtained a deeper structure theorem: The set of eigenvalues of $A$ with modulus $\rho(A)$ consists precisely of $\rho(A)$ times all the $m$th roots of unity, the spectrum $\sigma(A)$ of $A$ is invariant under a rotation about the origin of the complex plane through an angle of $2\pi/m$, and $A$ is an $m$-cyclic matrix, i.e., there is a permutation matrix $P$ such that $P^tAP$ is a matrix of the form

$$\begin{bmatrix} \mathbf{0} & A_{12} & & & \\ & \mathbf{0} & A_{23} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{0} & A_{m-1,m} \\ A_{m1} & & & & \mathbf{0} \end{bmatrix}, \tag{1.1}$$

where the zero blocks along the diagonal are all square. To establish the above structure theorem, the most popular proof is to use Wielandt's lemma. (Some relevant definitions and a full statement of Wielandt's lemma will be given later on. For the logical relations between the various conditions that appear in the above-mentioned Frobenius's result, in the setting of a complex matrix, see the recent paper [T2] by the second author.)

By now, the Perron-Frobenius theory of a nonnegative matrix is very well-known. Almost every textbook of matrix theory contains a chapter on the subject, and there are also monographs specially devoted to nonnegative matrices and their applications. On the other hand, there is not much literature on the numerical range of a nonnegative matrix, although it is well-known that the numerical range of a matrix and its spectral properties are related. As a matter of fact, as early as 1966, Issos [I] in his unpublished Ph.D. thesis has obtained some Perron-Frobenius type results on the numerical range of an irreducible nonnegative matrix. However, for many years, except for a reference by Fiedler [F], it appears that Issos's work was almost unnoticed. Recall that the numerical range of an $n$-by-$n$ complex matrix $A$ is denoted and defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

Here is the main result obtained by Issos [I, Theorem 7]:

**Theorem A.** *Let $A$ be an irreducible nonnegative matrix with index of imprimitivity $m$. Denote the numerical radius of $A$ by $w(A)$. Then*

$$\{\lambda \in W(A) : |\lambda| = w(A)\} = \{w(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\}.$$

Issos's proof depends on a number of auxiliary results and is rather tedious. Recently, the second author and Yang [TY] also obtained Issos's main result as a side-product of their treatment. The proof given in [TY, Corollary 2] for Issos's result may not be easily accessible to the general readers. This is because the proof is indirect, graph-theoretic, and depends on results from the previous paper [T2] of the second author, on the less well-known concepts of the signed length of a cycle (which is different from that of a circuit) and matrix cycle products, and also on a characterization of diagonal similarity between matrices in terms of matrix cycle products due to Saunders and Schneider. (Indeed, it is a purpose of the papers [T2] and [TY] to demonstrate the usefulness of these less well-known concepts and the characterization of Saunders and Schneider.) This research was initiated by our attempt to find a direct, self-contained proof of Issos's main result. In a graph-free manner, we are able to do this and also obtain an extension of the result in the setting of a nonnegative matrix with irreducible real part. Then, in terms of certain graph-theoretic concepts, we put the latter result in a more concrete usable form, depending our proof on some results of [T1, T2].

Since the literature on numerical range analogs of the Perron-Frobenius theory is scanty, we also think it is worthwhile to offer a complete and, as far as possible, self-contained development here.

The results we obtain also enable us to solve the open problem of characterizing nonnegative matrices whose numerical ranges are regular polygons with center at the origin. We treat this problem and related problems in the second half of the paper.

## 2 Preliminaries

We assume knowledge of the Perron-Frobenius theory of nonnegative matrices, which is available in many standard textbooks such as [BP], [HJ1], or [M], as well as familiarity with numerical ranges (see, for instance, [GR] or [HJ2]).

Below we give a list of notations which we will follow. We always use $A$ to denote an $n$-by-$n$ complex matrix for some fixed positive integer $n$.

| | |
|---|---|
| $M_n$ | the set of all $n$-by-$n$ complex matrices; |
| $\mathbb{R}_+^n$ | the nonnegative orthant of $\mathbb{R}^n$; |
| $W(A)$ | the (classical) numerical range of $A$; |
| $w(A)$ | the numerical radius of $A$; |
| $\sigma(A)$ | the spectrum of $A$; |
| $\rho(A)$ | the spectral radius of $A$; |
| $A^t$ | the transpose of $A$; |
| $A^*$ | the conjugate transpose of $A$; |
| $\operatorname{Re} A$ | the real part of $A$, i.e., $(A + A^*)/2$; |
| $\|A\|$ | the matrix $(\|a_{rs}\|)$ (where $A = (a_{rs})$) |
| $\operatorname{Re} z$ | the real part of $z$ (where $z$ is a complex number) |
| $\|x\|$ | the vector $(\|\xi_1\|, \ldots, \|\xi_n\|)^t$ (where $x = (\xi_1, \ldots, \xi_n)^t$) |
| $\lambda_{\max}(H)$ | the largest eigenvalue of $H$ (where $H$ is hermitian) |
| $i$ | the imaginary unit $\sqrt{-1}$; |
| $\langle n \rangle$ | the set $\{1, 2, \ldots, n\}$. |

For a vector $x \in \mathbb{C}^n$, we use $\|x\|$ to denote the Euclidean norm of $x$, i.e., $\|x\| = (x^*x)^{1/2}$. For a matrix $A$, we use $\|A\|$ to denote the operator norm of $A$, i.e., $\max_{\|x\|=1} \|Ax\|/\|x\|$.

For real matrices $A$, $B$ of the same size, we use $A \geq B$ (respectively, $A > B$) to mean $a_{rs} \geq b_{rs}$ (respectively, $a_{rs} > b_{rs}$) for all indices $r$, $s$. The notation will also apply to vectors.

We call a matrix $A \in M_n$ *irreducible* if $n = 1$, or $n \geq 2$ and there does not exist a permutation matrix $P$ such that

$$P^t A P = \begin{bmatrix} B & C \\ O & D \end{bmatrix},$$

where $B$, $D$ are nonempty square matrices.

Given $A$, $B \in M_n$, $A$ is said to be *diagonally similar* to $B$ if there exists a nonsingular diagonal matrix $D$ such that $A = D^{-1}BD$; if, in addition, $D$ can be chosen to be unitary, then we say $A$ is *unitarily diagonally similar* to $B$.

It is known [TY, Remarks 2 and 5] and not difficult to show the following:

**Remark 2.1.** For any $A \in M_n$ and any unit complex number $\xi$, we have:

(i) $A$ is unitarily diagonally similar to $\xi A$ if and only if $A$ is diagonally similar to $\xi A$;

(ii) $\operatorname{Re} A$ is unitarily diagonally similar to $\operatorname{Re}(\xi A)$ if and only if $\operatorname{Re} A$ is diagonally similar to $\operatorname{Re}(\xi A)$.

For graph-theoretic definitions, we follow those of [T2] and [TY]. We need, in particular, the concepts of cyclic index of a matrix, a cycle in a digraph, and the signed length of a cycle, which we are going to explain.

For any $A \in M_n$, as usual, by the *digraph of* $A$, denoted by $G(A)$, we mean the directed graph with vertex set $\langle n \rangle$ such that $(r, s)$ is an arc if and only if $a_{rs} \neq 0$. By the *undirected graph of* $A$ we mean the undirected graph obtained from $G(A)$ by removing the direction of its arcs. We call an undirected graph *connected* if either it has exactly one vertex or it has more than one vertex and every pair of distinct vertices are joined by a path.

It is well-known (see, for instance, [HJ1, Theorem 6.2.24]) that a matrix $A \in M_n$ is irreducible if and only if its digraph $G(A)$ is strongly connected (in the sense that given any two vertices $r$, $s$ of $G(A)$, there is a directed path in $G(A)$ from $r$ to $s$ and vice versa). It is not difficult to show the following:

**Remark 2.2.** For any $A \in M_n$, if $\operatorname{Re} A$ is irreducible, then the undirected graph of $A$ is connected. The converse also holds if $A$ is nonnegative.

We call a matrix $A \in M_n$ *$m$-cyclic* if there exists a permutation matrix $P$ such that $P^t AP$ is of the form (1.1) where the zero blocks along the diagonal are all square. The largest positive integer $m$ for which a matrix $A$ is $m$-cyclic is called the *cyclic index* of $A$. We call $A$ a *block-shift matrix* if for some integer $m \geq 2$, $A$ is of the form (1.1) and with $A_{m1} = 0$. An $m$-cyclic matrix (respectively, a matrix which is permutationally similar to a block-shift matrix) can be characterized as one whose digraph is cyclically $m$-partite (respectively, linearly partite) (see [T2] for definitions).

We reserve the term "circuit" (in a digraph) for its usual meaning, i.e., a simple closed directed path. For instance, a sequence of arcs, like (1,2), (2,3), (3,4), and (4,1), forms a circuit of length 4. The term "cycle" in our usage means something different. For example, a sequence of arcs, like

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longleftarrow 1,$$

forms a cycle of length 5 and signed length 1. Here we use $r \longrightarrow s$ to denote the arc $(r, s)$ traversed from $r$ to $s$ and referred to it as a positive link, and use $s \longleftarrow r$ to denote the arc $(r, s)$ traversed from $s$ to $r$ and referred to it as a negative link. The number of positive links minus the number of negative links in a cycle gives the signed length of the cycle. (For formal definitions, see [T2, Section 2].)

## 3 Numerical Range Analogs of the Perron-Frobenius Theory

Let $A$ be an $n$-by-$n$ nonnegative matrix. In parallel to the Perron-Frobenius theory, it is natural to assert that $w(A) \in W(A)$ and there is a unit nonnegative vector $x$ such that $x^*Ax = w(A)$. The assertion is, indeed, true and is also pretty obvious. The reason is, for any unit vector $z \in \mathbb{C}^n$, we have $|z^*Az| \leq |z|^*A|z|$; hence

$$w(A) = \sup\{|z^*Az| : \|z\| = 1\} = \sup\{y^tAy : \|y\| = 1,\ y \in \mathbb{R}^n_+\}.$$

Clearly, the continuous real-valued map $y \mapsto y^tAy$ attains its maximum on the intersection of the unit sphere with the nonnegative orthant $\mathbb{R}^n_+$, which is a compact set. Hence our assertion follows. (The foregoing discussion has essentially appeared in [I, Theorem 1 and its proof], where it is assumed that the matrix $A$ is irreducible nonnegative.)

The next thing one may try to prove is that, if $A$ is irreducible nonnegative, then there is a positive unit vector $x$ such that $x^*Ax = w(A)$, and furthermore $x$ is unique. It is desirable that we can somehow apply the Perron-Frobenius theory. The following general result enables us to do this.

**Lemma 3.1.** *Let $A \in M_n$ and let $\xi$ be a unit complex number such that $\xi w(A) \in W(A)$. Then*

(i) $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A)) = \rho(\mathrm{Re}(\bar{\xi}A)) = w(A)$;

(ii) *the set $V = \{x \in \mathbb{C}^n \colon x^*Ax = \xi w(A)\|x\|^2\}$ is equal to the eigenspace of $\mathrm{Re}(\bar{\xi}A)$ corresponding to $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$.*

*Proof.* (i) Since $\xi w(A) \in W(A)$, we can find a nonzero vector $u$ that satisfies $u^*Au = \xi w(A)\|u\|^2$. Then $u^*(\bar{\xi}A)u = w(A)\|u\|^2$, and so $u^*(\xi A^*)u = w(A)\|u\|^2$. Adding the two equations, we obtain $u^*\mathrm{Re}(\bar{\xi}A)u = w(A)\|u\|^2$ or $u^*(w(A)I_n - \mathrm{Re}(\bar{\xi}A))u = 0$. Note that the matrix $w(A)I_n - \mathrm{Re}(\bar{\xi}A)$ is positive semidefinite, as we have

$$w(A) = w(\bar{\xi}A) \geq w(\mathrm{Re}(\bar{\xi}A)) = \rho(\mathrm{Re}(\bar{\xi}A)) \geq \lambda_{\max}(\mathrm{Re}(\bar{\xi}A)).$$

Hence, $u$ is an eigenvector of $\mathrm{Re}(\bar{\xi}A)$ corresponding to $w(A)$, and also it follows that we have $w(A) = \rho(\mathrm{Re}(\bar{\xi}A)) = \lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$.

(ii) If $x$ is any nonzero vector in $V$, then by what we have done in the proof of part(i) (with $x$ in place of $u$), we see that $x$ is an eigenvector of $\mathrm{Re}(\bar{\xi}A)$ corresponding to $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$.

Conversely, if $x$ is an eigenvector of $\mathrm{Re}(\bar{\xi}A)$ corresponding to $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$ ($= w(A)$), then we have

$$w(A)\|x\|^2 = x^*\mathrm{Re}(\bar{\xi}A)x = \mathrm{Re}(x^*(\bar{\xi}A)x) \le |x^*(\bar{\xi}A)x| \le w(A)\|x\|^2,$$

and hence $x^*(\bar{\xi}A)x = w(A)\|x\|^2$, i.e., $x \in V$. □

Lemma 3.1(ii) is well known to researchers of numerical range. For example, it is essentially contained in [DW, Corollary 1.4], and is also partly a consequence of the following result in [E] (see also [GR, Theorems 1.5–1 and 1.5–2]):

*A point $\alpha \in W(A)$ is an extreme point if and only if the associated subset $\{x \in \mathbb{C}^n : x^*Ax = \alpha\|x\|^2\}$ is a linear subspace.*

We take a digression here. By examining the above proof of Lemma 3.1 (or the proof of [DW, Corollary 1.4]) carefully, one can see that our argument also shows the following:

**Remark 3.2.** Let $A \in M_n$. For any unit complex number $\xi$, the set $\{x \in \mathbb{C}^n : x^*Ax = \xi w(A)\|x\|^2\}$ is equal to the nullspace of $w(A)I_n - \mathrm{Re}(\bar{\xi}A)$. Consequently, $\xi w(A) \in W(A)$ if and only if $\det(w(A)I_n - \mathrm{Re}(\bar{\xi}A)) = 0$.

The last part of the above remark (the "only if" part of which is implicit in the proof of [TY, Lemma 6]) enables us to check whether a given nonnegative matrix $A$ with irreducible real part has a circular disk centered at origin as its numerical range, or whether it satisfies $e^{2\pi i/m}W(A) = W(A)$ for a given positive integer $m$. This is because, by [TY, Theorems 1 and 2], for such a matrix $A$, $W(A)$ is a circular disk centered at the origin if and only if for some real number $\theta$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p$, $q$ are relatively prime integers with $q > n$, we have $e^{i\theta}w(A) \in W(A)$; $e^{2\pi i/m}W(A) = W(A)$ if and only if $e^{2\pi i/m}w(A) \in W(A)$.

Now back to numerical range analogs of the Perron-Frobenius theory. If $A$ is a nonnegative matrix, we already know that $w(A) \in W(A)$. So in this case we can apply Lemma 3.1 to $A$ by taking $\xi = 1$. Then we see that we have

$$\lambda_{\max}(\mathrm{Re}\,A) = \rho(\mathrm{Re}\,A) = w(A),$$

and the set $\{x \in \mathbb{C}^n\colon x^*Ax = w(A)\|x\|^2\}$ is equal to the eigenspace of the nonnegative matrix $\mathrm{Re}\,A$ corresponding to its spectral radius $\rho(\mathrm{Re}\,A)$. If, in addition, $\mathrm{Re}\,A$ is irreducible (which is the case if $A$ is irreducible), then by the Perron-Frobenius theory, $\rho(\operatorname{Re} A)$ is a simple eigenvalue of $\operatorname{Re} A$ and the said subspace is of dimension 1, spanned by a positive vector.

Summarizing, we have obtained the following:

**Proposition 3.3.** *Let $A \in M_n$ be nonnegative. Then $w(A) \in W(A)$ and each of the following numbers is equal to $w(A)$ :*

$$\max\{y^t A y : y \in \mathbb{R}^n_+, \|y\| = 1\}, \ \lambda_{\max}(\operatorname{Re} A), \ \textit{and } \rho(\operatorname{Re} A).$$

*Moreover, there is a unit nonnegative vector $x$ such that $x^*Ax = w(A)$. If, in addition,* Re *$A$ is irreducible, then the vector $x$ is unique and is positive.*

The relation $w(A) = \rho(\operatorname{Re} A)$ for a nonnegative matrix $A$ was shown in [GT]. It also appeared implicitly in the proof of [I, Theorem 1].

We make another digression and take note of the following interesting consequence of the fact that $w(A) \in W(A)$ for a nonnegative matrix $A$:

**Corollary 3.4.** *Let $A = (a_{ij})$ be an $n$-by-$n$ nonnegative matrix. If $W(A)$ is a (possibly degenerate) elliptic disk or a regular polygon, then the center $p$ of $W(A)$ must be a nonnegative real number such that $p \geq \min_{1 \leq j \leq n} a_{jj}$.*

*Proof.* Since $A$ is a real matrix, $W(A)$ must be symmetric about the real axis (see, for instance, [NT, Lemma 3.1]). So $p$ must lie on the real axis. Let $\alpha$ denote $\min_{1 \leq j \leq n} a_{jj}$, and assume to the contrary that $p < \alpha$. Then $A - \alpha I_n$ is still a nonnegative matrix and its numerical range is $W(A) - \alpha$, with center at $p - \alpha$, which is a negative number. If $W(A)$ is an elliptic disk, then, clearly, the left vertical supporting line for $W(A - \alpha I_n)$ is farther away from the origin than the right vertical supporting line. Hence, $w(A - \alpha I_n) \notin W(A - \alpha I_n)$, which contradicts the result of Proposition 3.3. On the other hand, if $W(A)$ is a regular polygon, then the distance from the origin to the vertices of $W(A - \alpha I_n)$ other than $w(A) - \alpha$ is greater that $w(A) - \alpha$, which again contradicts Proposition 3.3. □

The above corollary may suggest that, in general, if $A = (a_{ij})$ is an $n$-by-$n$ nonnegative matrix, then the centroid $p$ of $W(A)$ satisfies $p \geq \min_{1 \leq j \leq n} a_{jj}$. Our next example will show that this is not true.

**Example 3.5.** Consider the nonnegative matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $W(A)$ is the convex hull of the equilateral triangle with vertices 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$ and the line segment with endpoints 1 and $-1$. So we have $\min_{1\leq j\leq n} a_{jj} = 0 > p$, where $p$ denotes the centroid of $W(A)$. By perturbing the above $A$ a little bit, one can also give an irreducible nonnegative matrix as an example.

Next, we turn to a comparison of $w(A)$ and $w(B)$ for nonnegative matrices $A$ and $B$ with $A \geq B$. For spectral radius, the following is well known (see [M, p. 38, Corollary 2.2]):

*Let $A,\ B \in M_n$ be nonnegative, and suppose that $B \leq A$. Then $\rho(B) \leq \rho(A)$. If, in addition, $A$ is irreducible and $A \neq B$, then $\rho(B) < \rho(A)$.*

Using the relation $w(A) = \rho(\operatorname{Re} A)$ for a nonnegative matrix $A$, we immediately obtain the following corresponding result for numerical radius. Below we also give an alternative short proof of the result.

**Corollary 3.6.** *Let $A,\ B \in M_n$ be nonnegative, and suppose that $B \leq A$. Then $w(B) \leq w(A)$. If, in addition,* $\operatorname{Re} A$ *is irreducible and $A \neq B$, then $w(B) < w(A)$.*

*Proof.* Since $0 \leq B \leq A$, we have

$$\begin{aligned} w(B) &= \max\{x^t B x : x \in \mathbb{R}^n_+,\ \|x\| = 1\} \\ &\leq \max\{x^t A x : x \in \mathbb{R}^n_+,\ \|x\| = 1\} \\ &= w(A). \end{aligned}$$

Now assume that $\operatorname{Re} A$ is irreducible, and suppose that $w(B) = w(A)$. Choose a nonnegative unit vector $x$ such that $x^* B x = w(B)$. Then we have

$$w(A) = w(B) = x^* B x \leq x^* A x \leq w(A).$$

Thus, the two inequalities become equalities. Since $\operatorname{Re} A$ is irreducible, by the last part of Proposition 3.3, the vector $x$ is positive. So we have $x^*(A - B)x = 0$, and together with the assumption $A \geq B$, it follows that $A = B$, which is a contradiction. □

We want to emphasize that in the last part of Corollary 3.6 (also Proposition 3.3) we are assuming that $\operatorname{Re} A$ is irreducible instead of $A$ being irreducible. And this is the right setting for results on numerical radius. For the corresponding results on spectral radius, we do need the irreducibility assumption. As an example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\operatorname{Re} A$ is irreducible, but not $A$, and we have $\rho(A) = \rho(B) = 0$, and $w(A) > w(B)$.

In the above we have treated the rudimentary part of the numerical range analogs of the Perron-Frobenius theory. To proceed further, we need the following Wielandt's lemma [W]:

**Wielandt's Lemma.** *Let $A,\ B \in M_n$, and assume that $A$ is nonnegative. If $|B| \le A$, then $\rho(B) \le \rho(A)$. Assume, in addition, that $A$ is irreducible. If $\rho(A) = \rho(B)$ and $\xi$ is a unit complex number such that $\xi\rho(B) \in \sigma(B)$, then $B = \xi DAD^{-1}$ for some (unitary) diagonal matrix $D$.*

In the above formulation of Wielandt's lemma, in order to emphasize its nontrivial part, we have deliberately omitted the obvious converse part for its second half (cf. [M, p. 36, Theorem 2.1]).

Now consider an irreducible nonnegative matrix $A$ with index of imprimitivity $m$. By the Perron-Frobenius theory, we have $e^{2\pi i/m}\rho(A) \in \sigma(A)$, and so by the second half of Wielandt's lemma (with $B = A$ and $\xi = e^{2\pi i/m}$), $e^{2\pi i/m}A$ is (unitarily) diagonally similar to $A$. But the numerical range of a matrix is unchanged if we apply a unitary similarity to the matrix, hence we have $e^{2\pi i/m}W(A) = W(e^{2\pi i/m}A) = W(A)$, i.e., $W(A)$ is invariant under a rotation about the origin of the complex plane through an angle of $2\pi/m$. But we also have $w(A) \in W(A)$, hence $w(A)e^{2\pi ti/m} \in W(A)$ for $t = 0, 1, \ldots, m-1$. This proves the easy half of Theorem A. (An argument almost the same as the preceding one can be found in [I, Theorems 6 and 7], except that Issos used the $m$-cyclicity of $A$ to deduce the diagonal similarity between $A$ and $e^{2\pi i/m}A$ instead of applying Wielandt's lemma.) To prove the reverse inclusion, we need the following:

**Proposition 3.7.** *Let $A \in M_n$ be nonnegative, and suppose* Re *$A$ is irreducible. If $\xi$ is a unit complex number such that $\xi w(A) \in W(A)$, then $DAD^{-1} = \xi A$ for some unitary diagonal matrix $D$.*

To see how Proposition 3.7 can be used to establish the remaining inclusion for Theorem A, consider any unit complex number $\xi$ for which $\xi w(A) \in W(A)$. By the proposition, $\xi A$ is similar to $A$. But $\rho(A)$ is an eigenvalue of $A$, hence so is $\xi\rho(A)$. By the Frobenius theorem for an irreducible nonnegative matrix, it follows that $\xi$ must be an $m$th root of unity, where $m$ is the index of imprimitivity of $A$. This completes the proof of Theorem A.

Note that when $A$ is an irreducible nonnegative matrix with index of imprimitivity 1 (or, equivalently, when it is a primitive matrix, i.e., a nonnegative matrix, one of whose powers is positive), Theorem A tells us that the numerical range of

$A$ contains exactly one point with modulus $w(A)$, namely, $w(A)$ itself. Our above proof also covers this special case.

Theorem A first appeared in [I, Theorem 7] and then in [TY, Corollary 2]; whereas Proposition 3.7 is contained in [TY, Lemma 1], but not in [I]. The proof given in [TY] for Theorem A is longer than necessary; it makes use of [TY, Lemma 1], but not in the best way. Graph-theoretic arguments as well as results from [T2] are needed in [TY] to establish its Lemma 1. We shall give two proofs for Proposition 3.7, which are self-contained and graph-free.

*First Proof of Proposition* 3.7: Since $A$ is nonnegative, we have $\rho(\operatorname{Re} A) = w(A)$. By Lemma 3.1(i), we also have $w(A) = \rho(\operatorname{Re}(\bar{\xi}A)) = \lambda_{\max}(\operatorname{Re}(\bar{\xi}A))$, and hence $\rho(\operatorname{Re}(\bar{\xi}A)) = \rho(\operatorname{Re} A)$. From the above, it is also clear that $\rho(\operatorname{Re}(\bar{\xi}A))$ is an eigenvalue of $\operatorname{Re}(\bar{\xi}A)$. In view of $|\operatorname{Re}(\bar{\xi}A)| \leq \operatorname{Re} A$ and the irreducibility of $\operatorname{Re} A$, by the second half of Wielandt's lemma, it follows that there is a unitary diagonal matrix $D$, say, $D = \operatorname{diag}(d_1, \ldots, d_n)$, such that $D(\operatorname{Re}(\bar{\xi}A))D^{-1} = \operatorname{Re} A$, i.e., $D(\bar{\xi}A + \xi A^t)D^{-1} = A + A^t$. By equating the corresponding entries of both sides, we obtain $d_r(\bar{\xi}a_{rs} + \xi a_{sr})d_s^{-1} = a_{rs} + a_{sr}$ for all $r,\ s \in \langle n \rangle$. Since $|d_r\bar{\xi}a_{rs}d_s^{-1}| = a_{rs}$ and $|d_r\xi a_{sr}d_s^{-1}| = a_{sr}$ (as $|d_r| = |d_s| = |\xi| = 1$), it follows that we have $d_r\bar{\xi}a_{rs}d_s^{-1} = a_{rs}$ (and $d_r\xi a_{sr}d_s^{-1} = a_{sr}$) for all $r,\ s \in \langle n \rangle$. Hence, we have $D(\bar{\xi}A)D^{-1} = A$, or $DAD^{-1} = \xi A$. □

Our second proof of Proposition 3.7 will depend on the following numerical radius analog of Wielandt's lemma, which is of independent interest.

**Lemma 3.8.** *Let $A,\ B \in M_n$, and assume that $A$ is nonnegative. If $|B| \leq A$, then $w(B) \leq w(A)$. Suppose, in addition, that* $\operatorname{Re} A$ *is irreducible. If $w(A) = w(B)$ and $\xi$ is a unit complex number such that $\xi w(A) \in W(B)$, then $B = \xi DAD^{-1}$ for some unitary diagonal matrix $D$.*

*Proof.* The first half of this lemma can be readily proved by modifying the argument given in the proof for the first half of Corollary 3.6. Alternatively, apply the first part of Wielandt's lemma to the pair of matrices $\operatorname{Re}(e^{i\theta}B)$, $\operatorname{Re} A$ (where $\theta \in \mathbb{R}$), and use the fact that for any matrix $A$, we have

$$w(A) = \max\{\lambda_{\max}(\operatorname{Re}(e^{i\theta}A)) : \theta \in \mathbb{R}\} = \max\{\rho(\operatorname{Re}(e^{i\theta}A)) : \theta \in \mathbb{R}\}.$$

The proof for the second half of this lemma runs parallel to a known proof for the corresponding part of Wielandt's lemma (cf. [M, pp. 37–38]). Let $y$ be a unit vector such that $y^*By = \xi w(A)$. Then

$$w(A) = y^*(\bar{\xi}B)y \leq |y|^t|B||y| \leq |y|^tA|y| \leq w(A);$$

hence, the above inequalities all become equalities. Since $|y|^t A|y| = w(A)$ and $\operatorname{Re} A$ is irreducible, by the last part of Proposition 3.3, we have $|y| > 0$. Now, in view of

$$|y|^t(A - |B|)|y| = w(A) - w(A) = 0, \ A - |B| \geq 0 \text{ and } |y| > 0,$$

we have $|B| = A$. Let $D$ denote the unitary diagonal matrix $\operatorname{diag}(\eta_1/|\eta_1|, \ldots, \eta_n/|\eta_n|)$, where $y = (\eta_1, \ldots, \eta_n)^t$. Then we have

$$|y|^t D^*(\bar{\xi}B)D|y| = y^*(\bar{\xi}B)y = w(A),$$

where the second equality has already been established above. But we also already have $|y|^t A|y| = w(A)$, so

$$|y|^t D^*(\bar{\xi}B)D|y| = |y|^t A|y|.$$

And since $|D^*(\bar{\xi}B)D| = |B| = A$ and $|y| > 0$, it follows that $\bar{\xi}D^{-1}BD = A$, or $B = \xi DAD^{-1}$. □

*Second Proof of Proposition* 3.7: Apply Lemma 3.8 with $B = A$. □

It is easy to show that for any $A \in M_n$ and any unit complex number $\xi$, if $A$ is (unitarily) diagonally similar to $\xi A$, then $\operatorname{Re} A$ is also (unitarily) diagonally similar to $\operatorname{Re}(\xi A)$ (see [TY, Remarks 2,4,5]). The converse is not true in general. In [TY, Lemma 2] it is shown that the converse is true if we assume, in addition, that the entries of $A$ satisfy $a_{rs}a_{sr} = 0$ for all $r, \ s \in \langle n \rangle$. In view of the argument given in the last part of the first proof of Proposition 3.7, we now have another situation when the converse is true:

**Remark 3.9.** Let $A$ be a nonnegative matrix. For any unit complex number $\xi$, $\xi A$ is diagonally similar to $A$ if and only if $\operatorname{Re}(\xi A)$ is diagonally similar to $\operatorname{Re} A$.

The above remark is actually implicit in the work of [TY]. This is because, by [TY, Lemma 3], we readily obtain our remark under the additional assumption that $\operatorname{Re} A$ is irreducible (as $\lambda_{\max}(\operatorname{Re}(\xi A)) = \lambda_{\max}(\operatorname{Re} A)$ whenever $\operatorname{Re}(\xi A)$ is similar to $\operatorname{Re} A$), and then after a simple calculation we can drop the additional assumption. Certainly, our present proof is more direct and easier.

For more equivalent conditions for $\xi w(A) \in W(A)$ (when $A$ is nonnegative and $|\xi| = 1$), see [TY, Lemma 3].

We would like to make another observation.

**Corollary 3.10.** *Let $A$ be a nonnegative matrix with irreducible real part. Let $\xi$ be a unit complex number such that $\xi w(A) \in W(A)$. Then the subspace $\{x \in \mathbb{C}^n : x^*Ax = \xi w(A)\|x\|^2\}$ is of dimension* 1.

*Proof.* By Lemma 3.1, the set $\{x \in \mathbb{C}^n : x^*Ax = \xi w(A)\|x^2\|\}$ is equal to the eigenspace of $\mathrm{Re}(\bar{\xi}A)$ corresponding to $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$ $(= w(A))$. By the first proof of Proposition 3.7 or by the proposition itself (and Remark 3.9), $\mathrm{Re}(\bar{\xi}A)$ is diagonally similar to $\mathrm{Re}\, A$. Since $\mathrm{Re}\, A$ is irreducible nonnegative, by the Perron-Frobenius theory, $\rho(\mathrm{Re}\, A)$ $(= w(A))$ is a simple eigenvalue of $\mathrm{Re}\, A$. Hence, $\lambda_{\max}(\mathrm{Re}(\bar{\xi}A))$ is also a simple eigenvalue of $\mathrm{Re}(\bar{\xi}A)$, and the said subspace is of dimension 1. □

We are going to extend Theorem $A$ to the case when $A$ is a nonnegative matrix with irreducible real part.

For any $A \in M_n$, it is easy to verify that the set

$$H = \{\xi \in \mathbb{C} : |\xi| = 1,\ \xi A \text{ is (unitarily) diagonally similar to } A\}$$

forms a subgroup of the group of all unit complex numbers, and moreover it is included in the set $\{\xi \in \mathbb{C} : |\xi| = 1,\ \xi W(A) = W(A)\}$. If $A$ is nonnegative, then since $w(A) \in W(A)$, the latter set, in turn, is included in $\{\xi \in \mathbb{C} : |\xi| = 1,\ \xi w(A) \in W(A)\}$. Now assume, in addition, that $\mathrm{Re}\, A$ is irreducible. Then, in view of Proposition 3.7, the three sets are all equal. The group $H$ may be infinite or finite. If $H$ is infinite or has more than $n$ elements, then the numerical range of $A$ contains more than $n$ points with modulus equal to $w(A)$. In this case, by a known result due to Anderson (see, for instance, [TY, Lemma 6]), $W(A)$ is equal to the circular disk with center at the origin and radius $w(A)$. Hence, $H$ is precisely the group of all unit complex numbers. On the other hand, if $H$ is a finite group, say with order $m$ $(\leq n)$, then by Lagrange's theorem in group theory, for any $\xi \in H$, we have $\xi^m = 1$, i.e., each element of $H$ is an $m$th root of unity. But the cardinality of $H$ is $m$, so it follows that $H$ is precisely the group of all $m$th roots of unity. Summarizing, we have, in fact, established the following:

**Proposition 3.11.** *Let $A$ be a nonnegative matrix with irreducible real part.*
(i) *For any unit complex number $\xi$, the following conditions are equivalent:*
(a) *$\xi A$ is diagonally similar to $A$.*
(b) $\xi W(A) = W(A)$.
(c) $\xi w(A) \in W(A)$.

(ii) *The set $\{\xi \in \mathbb{C} : |\xi| = 1,\ \xi w(A) \in W(A)\}$ is a group under multiplication, and is either the group of all unit complex numbers or is a finite (necessarily cyclic) subgroup of it.*

(iii) *If $W(A)$ is not a circular disk with center at the origin, then*

$$\{z \in W(A) : |z| = w(A)\} = \{w(A)e^{2\pi t i/m} : t = 0, 1, \ldots, m-1\},$$

*where $m$ is the largest positive integer such that $A$ is diagonally similar to $e^{2\pi i/m}A$.*

So far we are graph-free. Next, in terms of certain graph-theoretic concepts, we are going to rewrite part (iii) of Proposition 3.11 in a readily usable form.

In [T1, Theorem 1], the second author gave equivalent conditions on a complex matrix $A$ with the property that the numerical range of any matrix with the same digraph as $A$ is a circular disk centered at the origin. One equivalent condition is that $A$ is permutationally similar to a block-shift matrix. Another equivalent condition is that all cycles of $G(A)$ have zero signed length. In [TY, Theorem 1], a long list of further new equivalent conditions were added. In particular, rather unexpectedly, it was found that in the case when $A$ is nonnegative and has a connected undirected graph (or equivalently, with irreducible real part), the condition that $W(A)$ is a circular disk centered at the origin is also an equivalent condition.

On the other hand, by [T2, Theorem 4.1], for any $A \in M_n$ and any positive integer $k$, if $A$ is $k$-cyclic, then $A$ is diagonally similar to $e^{2\pi i/k}A$; if, in addition, the digraph $G(A)$ has at least one cycle with nonzero signed length, then the converse also holds.

In view of the above (and Remark 2.2), we can now rewrite Proposition 3.11 (iii) as follows:

**Theorem 3.12.** *Let $A$ be a nonnegative matrix with connected undirected graph. Suppose that the digraph $G(A)$ has at least one cycle with nonzero signed length. Then*

$$\{z \in W(A) : |z| = w(A)\} = \{w(A)e^{2\pi ti/m} : m = 0, 1, \ldots, m-1\},$$

*where $m$ is the cyclic index of $A$.*

By [T2, Corollary 4.2 (i)], when the digraph $G(A)$ has at least one cycle with nonzero signed length ($A$ not necessarily nonnegative), the cyclic index of $A$ is equal to the greatest common divisor of the signed lengths of the cycles in $G(A)$. So, the cyclic index $m$ considered in Theorem 3.12 can be determined.

We have already offered a self-contained proof (via Proposition 3.7) for Theorem A. Now let us show that Theorem $A$ can be recovered also from Theorem 3.12: If $A$ is irreducible, then, by part (iii) of the above-mentioned corollary of [T2], the cyclic index of $A$ is also equal to the greatest common divisor of the circuit lengths of $G(A)$. But it is well-known (see, for instance, [BP, p. 35, Theorem 2.30]) that the index of imprimitivity of an irreducible nonnegative matrix is equal to the greatest common divisor of the circuit lengths of its associated digraph. And, of course, the digraph of an irreducible matrix, being strongly connected, has at least one cycle with nonzero signed length (as every circuit can be regarded as a cycle with signed length equal to its length). Hence, we can recover Theorem $A$ from Theorem 3.12.

More generally, we have the following:

**Remark 3.13.** Let $A$ be a nonnegative matrix whose digraph has at least one cycle with nonzero signed length. Suppose $A$ is permutationally similar to $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are nonnegative matrices each with connected undirected graph. Then:

(i) The cyclic index of $A$ equals the greatest common divisor of the cyclic indices of those $A_j$ whose digraphs have cycles with nonzero signed lengths.

(ii) The set $\{\xi \in \mathbb{C} : |\xi| = 1, \ \xi w(A) \in W(A)\}$ is equal to $\bigcup_j \{\xi \in \mathbb{C} : |\xi| = 1, \ \xi w(A_j) \in W(A_j)\}$, where the union is taken over all $j$ for which $w(A_j) = w(A)$. If there is at least one $j$ for which $w(A_j) = w(A)$ and the digraph $G(A_j)$ has no cycles with nonzero signed length, then $W(A)$ is a circular disk and, consequently, the above set is precisely the group of all unit complex numbers. Otherwise, the set is a union of certain $Z_p$'s, where $Z_p$ denotes the group of all complex $p$th roots of unity, and moreover it always includes the set $\{w(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\}$, where $m$ is the cyclic index of $A$.

## 4 Nonnegative Matrices whose Numerical Ranges are Regular Polygons

In his thesis [I, p. 24] Issos asked the question of when the numerical range of an irreducible nonnegative matrix is a regular (convex) polygon (with center not necessarily at the origin). In [TY, Problem 2] Tam and Yang also posed the problem of characterizing nonnegative matrices whose numerical ranges are regular polygons with center at the origin. In this section, we are going to treat these problems.

A point $\alpha$ lying on the boundary of $W(A)$ is called a *sharp point* of $W(A)$ if $W(A)$ is included in an angular sector with apex at $\alpha$ and angle less than $\pi$. For a nonnegative matrix $A$, if $W(A)$ is a polygon, then $w(A)$ (being an extreme point) is necessarily one of the vertices and hence is a sharp point of $W(A)$. The problem of characterizing when $w(A)$ is a sharp point of $W(A)$ for a nonnegative matrix $A$ has been solved by Tam and Yang [TY, Theorem 4]. But we are going to rederive the result in a different way, relying ourselves on a general result about radial matrices. As the reader will see, our present approach has the merit that it gives us a better understanding, throws light on a known result in [NT] and in addition yields more new results.

A matrix $A \in M_n$ is called *spectral* if $\rho(A) = w(A)$; $A$ is *radial* if $\|A\| = \rho(A)$ or, equivalently, $w(A) = \|A\|$. (For equivalent conditions on a radial matrix or a spectral matrix, see [HJ2, p. 45, Problem 27; pp. 61–62, Problem 37].)

**Proposition 4.1.** *Let $A \in M_n$ be a radial matrix. Then:*

(i) *There exists a unitary matrix $U \in M_n$ such that $U^*AU = D \oplus B$, where $D$ is a diagonal matrix each of whose diagonal entries is of modulus $w(A)$ and $B$ is a (possibly empty) matrix that satisfies $w(B) < w(A)$.*

(ii) *$W(A)$ is the convex hull of the polygon whose vertices are all the points in $W(A)$ with modulus $w(A)$ and a (possible empty) compact convex set, included in the open circular disk centered at the origin with radius $w(A)$.*

(iii) *Every point $z$ in $W(A)$ with modulus $w(A)$ is a sharp point.*

*Proof.* Part (i) was proved in [P]. Part (ii) follows from the fact that $W(A) = \operatorname{conv}(W(D) \bigcup W(B))$, where $W(B)$ is in the open disk centered at the origin with radius $w(A)$. Part (ii) follows readily from (iii). □

Concerning the problem of characterizing when $w(A)$ is a sharp point, we treat the case of a nonnegative matrix with irreducible real part first. The following result is a strengthening of [TY, Remark 16]. We give an independent proof.

**Theorem 4.2.** *Consider the following conditions for a nonnegative matrix $A$:*

(a) *$A$ is radial.*

(b) *$w(A)$ is a sharp point of $W(A)$.*

(c) *$A$ is spectral.*

(d) $\rho(A) = \rho(\operatorname{Re} A)$.

(e) *$A$ and $A^t$ have a common nonnegative eigenvector corresponding to $\rho(A)$.*

(i) *We always have the implications* (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longleftrightarrow$ (d) $\Longrightarrow$ (e).

(ii) *When* $\operatorname{Re} A$ *is irreducible, conditions* (a)–(e) *are all equivalent.*

(iii) *If* $\operatorname{Re} A$ *is irreducible and conditions* (a)–(e) *are all satisfied, then $A$ is necessarily irreducible.*

*Proof.* (i) (a)$\Longrightarrow$(b): This follows from Proposition 4.1 (ii), as we always have $w(A) \in W(A)$ for a nonnegative matrix.

(b)$\Longrightarrow$(c): This follows from a result of Kippenhahn [HJ2, Theorem 1.6.3], which says that if $\alpha$ is a sharp point of $W(A)$, where $A$ is any complex matrix, then $\alpha$ must be an eigenvalue of $A$.

The equivalence of (c) and (d) follows from the relation $w(A) = \rho(\operatorname{Re} A)$ for a nonnegative matrix $A$.

The implication (c)$\Longrightarrow$(e) can be deduced from known results about normal eigenvalues; see [HJ2, Theorem 1.6.6 and Section 1.6, Problem 11]. To make this proof self-contained, we offer an argument: Let $x$ be a unit nonnegative eigenvector of $A$ corresponding to $\rho(A)$. Then

$$x^t(\operatorname{Re} A)x = x^t A x = \rho(A) = w(A).$$

Since $A$ is nonnegative, $w(A)$ is also equal to $\lambda_{\max}(\operatorname{Re} A)$. This, together with our choice of $x$, implies that $x$ is the desired common nonnegative eigenvector of $A$ and $A^t$ corresponding to $\rho(A)$ $(= \lambda_{\max}(\operatorname{Re} A))$.

(ii) In view of part(i), it suffices to show that when $\operatorname{Re} A$ is irreducible, we have the implication (e)$\Longrightarrow$(a). Let $x$ be a common nonnegative eigenvector of $A$ and $A^t$ corresponding to $\rho(A)$. Then clearly $x$ is also a nonnegative eigenvector of the irreducible nonnegative matrix $\operatorname{Re} A$ (corresponding to $\rho(A)$). As such, $x$ must be a positive vector (see, for instance, [M, p. 7, Theorem 2.2]). But we also have $A^tAx = \rho(A)^2 x$, and it is well-known that a positive eigenvector of a nonnegative matrix must correspond to its spectral radius (see, for instance, [HJ1, Corollary 8.1.30]), so $\rho(A^tA) = \rho(A)^2$. Hence we have $\|A\|^2 = \rho(A^tA) = \rho(A)^2$, or $\|A\| = \rho(A)$, i.e., $A$ is radial.

(iii) It suffices to show that if $A$ is a nonnegative matrix with irreducible real part, and if $A$ is spectral, then $A$ is irreducible. We assume to the contrary that $A$ is reducible. By applying a permutation similarity, we may assume that $A$ is already in the Frobenius normal form, i.e., a (lower) triangular block form with, say, $p$ irreducible blocks $A_{11}, \ldots, A_{pp}$ along the diagonal (see, for instance, [BP, p. 39]). Let $B$ denote the matrix $A_{11} \oplus \cdots \oplus A_{pp}$. Then we have $\rho(A) = \max_{1 \le j \le p} \rho(A_j) = \rho(B)$. Since $A$ is reducible and the undirected graph of $A$ is connected, clearly we have $p \ge 2$, $A \ge B$ and $A \ne B$. By Corollary 3.6, it follows that we have $w(A) > w(B) \ge \rho(B) = \rho(A)$, which is a contradiction. □

Our next example will show that, for a general nonnegative matrix $A$, the missing implications in Theorem 4.2 (i) do not hold in general.

**Example 4.3.** Consider the nonnegative matrix $A = A_1 \oplus A_2$, with $A_1 = [1]$ and $A_2 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, where $\alpha$ is a positive number to be chosen. Note that we have $\rho(A) = 1$, $w(A) = \max\{1, \alpha/2\}$ and $\|A\| = \max\{1, \alpha\}$. Moreover, $W(A_1) = \{1\}$, $W(A_2)$ is the circular disk centered at the origin with radius $\alpha/2$, and $W(A) = \operatorname{conv}(W(A_1) \bigcup W(A_2))$. It is clear that $A$ and $A^t$ always have a common nonnegative eigenvector corresponding to $\rho(A)$ $(= 1)$, namely, the vector

$(1,0,0)^t$. If $\alpha > 2$, then $w(A) = \alpha/2 > \rho(A)$, and so $A$ is not spectral. This shows that for the conditions (a)–(e) of Theorem 4.2, (e)$\not\Rightarrow$(c). If $\alpha = 2$, then $\rho(A) = w(A) = 1$ and so $A$ is spectral. However, in this case, $W(A)$ is the circular disk centered at the origin with radius $w(A)$, and $w(A)$ is not a sharp point of $W(A)$. This shows that (c)$\not\Rightarrow$(b). Finally, if $1 < \alpha < 2$, then clearly $w(A)$ is a sharp point of $W(A)$. Since $\|A\| = \alpha > 1 = \rho(A)$, $A$ is not radial. This shows that (b)$\not\Rightarrow$(a).

It is clear that we can use Theorem 4.2 (and an argument given in the first paragraph of the proof of [TY, Lemma 5]) to recover [TY, Lemma 5]. Now we use Theorem 4.2 to derive [TY, Theorem 4]:

**Corollary 4.4.** *Let $A$ be a nonnegative matrix, and suppose $A$ is permutationally similar to $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are nonnegative matrices each with connected undirected graph. A necessary and sufficient condition for $w(A)$ to be a sharp point of $W(A)$ is that, for any $j$, $1 \le j \le k$, we have*

(a) *If $\rho(A_j) = \rho(A)$, then $A_j$ is itself an irreducible matrix and $\rho(A_j) = w(A_j)$.*

(b) *If $\rho(A_j) < \rho(A)$, then $w(A_j) < \rho(A)$.*

*Proof.* "Necessity": Consider any $j \in \langle k \rangle$ for which $\rho(A_j) = \rho(A)$. Since $w(A)$ is a sharp point of $W(A)$, we have $\rho(A) = w(A) \ge w(A_j) \ge \rho(A_j) = \rho(A)$, hence $w(A_j) = \rho(A_j)$, i.e., $A_j$ is spectral. Since $\operatorname{Re} A_j$ is irreducible, by Theorem 4.2 (ii) and (iii), it follows that $A_j$ is irreducible.

Now consider any $j \in \langle k \rangle$ for which $\rho(A_j) < \rho(A)$. If $w(A_j) = \rho(A)$, then we have $\rho(A_j) < w(A_j)$, and so $w(A)$ $(= w(A_j))$ is not a sharp point of $W(A_j)$, and hence also not a sharp point of $W(A)$, which is a contradiction.

"Sufficiency": When conditions (a) and (b) are satisfied, clearly we have $w(A) = \max_{1 \le j \le k} w(A_j) = \rho(A)$. For any $j$ for which $\rho(A_j) = \rho(A)$, by condition (a) and Theorem 4.2, $w(A_j)$ is a sharp point of $W(A_j)$ and the matrix $A_j$ is radial. Moreover, by Proposition 4.1 (ii), for any such $j$, $W(A_j)$ is the convex hull of $w(A)$ $(= w(A_j))$ and some compact convex set $C_j$ not containing $w(A)$. On the other hand, if $j$ is such that $\rho(A_j) < \rho(A)$, then, by condition (b), $W(A_j)$ is a compact convex set not containing $w(A)$ $(= \rho(A) > w(A_j))$. It is clear that the convex hull of all $C_j$ for which $\rho(A_j) = \rho(A)$ and all $W(A_j)$ for which $\rho(A_j) < \rho(A)$ is a compact convex set $C$ that does not contain $w(A)$. But $W(A)$ is the convex hull of $w(A)$ and $C$, hence $w(A)$ is a sharp point of $W(A)$. □

In [NT, Theorem 1.2] Nylen and Tam proved that if $A$ is a primitive doubly stochastic matrix, then $W(A)$ is symmetric about the real axis and is the convex hull of the point 1 and a compact convex set included in the open unit disk. Motivated by their result, we have the following for a nonnegative matrix:

**Proposition 4.5.** *Let $A$ be a nonnegative matrix, and suppose $A$ is permutationally similar to $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are nonnegative matrices each with irreducible real part. If $w(A)$ is a sharp point of $W(A)$, then we have $W(A) = \mathrm{conv}(P \bigcup C)$, where $P$ is the polygon with vertices consisting of all points in $W(A)$ with modulus $w(A)$, and $C$ is some compact convex set included in the open circular disk centered at the origin with radius $w(A)$.*

*Proof.* Since $w(A)$ is a sharp point of $W(A)$, conditions (a) and (b) of Corollary 4.4 are fulfilled. Consider any $j \in \langle k \rangle$. If $\rho(A_j) < \rho(A)$, then $W(A_j)$ is a compact convex set included in the open circular disk centered at the origin with radius $w(A)$. If $\rho(A_j) = \rho(A)$, then $w(A_j)$ is a sharp point of $W(A_j)$ (see the "necessity part" of the proof of Corollary 4.4), and by Theorem 4.2 (ii), the matrix $A_j$ is radial. In this case, by Proposition 4.1 (ii), $W(A_j)$ is the convex hull of a polygon with vertices all of modulus $w(A_j)$ $(= w(A))$ and some compact convex set included in the open circular disk centered at the origin with radius $w(A)$. In view of $W(A) = \mathrm{conv}(W(A_1) \bigcup \cdots \bigcup W(A_k))$, it is ready to see that our assertion follows.□

**Corollary 4.6.** *Let $A$ be a primitive matrix. If $A$ satisfies one of the conditions* (a)–(e) *in Theorem 4.2, then $W(A)$ is symmetric about the real axis and is the convex hull of the point $\rho(A)$ and a compact convex set included in the open circular disk centered at the origin with radius $\rho(A)$.*

In view of Theorem 4.2 (iii) and the following result, in solving the problem of characterizing nonnegative matrices whose numerical ranges are regular polygons with center at the origin, we may focus our attention to irreducible nonnegative matrices.

**Theorem 4.7.** *Let $A$ be a nonnegative matrix. Suppose $A$ is permutationally similar to $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are nonnegative matrices each with irreducible real part. Then $W(A)$ is a regular polygon with center at the origin if and only if there exists $s \in \langle k \rangle$ such that $W(A_s)$ is a regular polygon with center at the origin, and for every $j \in \langle k \rangle$, $j \neq s$, we have $W(A_j) \subseteq W(A_s)$.*

*Proof.* The "if" part is obvious. Since $w(A)$ is always an extreme point of $W(A)$ (as $A$ is nonnegative), to prove the "only if" part, we may suppose that $W(A)$ is the regular polygon with center at the origin given by

$$W(A) = \mathrm{conv}\left\{w(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\right\} \quad \text{for some } m \geq 2.$$

By our assumption on $A$, clearly, $W(A) = \operatorname{conv}(W(A_1) \bigcup \cdots \bigcup W(A_k))$. But $w(A)e^{2\pi i/m}$ is an extreme point of $W(A)$, so it must belong to one of the sets $W(A_1), \ldots, W(A_k)$, say, $W(A_s)$. Then $w(A_s) = w(A)$ and, by Proposition 3.11 (ii), $W(A_s)$ contains each of the points $w(A)e^{2\pi ti/m}$, $t = 0, 1, \ldots, m-1$. Hence, by the convexity of the numerical range of a matrix, we have

$$W(A_s) \supseteq \operatorname{conv}\{w(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\} = W(A).$$

Certainly, we also have $W(A_s) \subseteq W(A)$. So our assertion follows. □

We would like to mention that a similar result also holds for the question of when a general complex matrix has a circular disk with center at the origin as its numerical range (see [TY, Theorem 3]). We also want to emphasize that in Theorem 4.7 the nonnegativity assumption on $A$ cannot be dropped. Counterexamples can be easily constructed.

An application of Theorem 3.12 yields the following related result:

**Proposition 4.8.** *Let $A$ be a nonnegative matrix with connected undirected graph. Suppose that the digraph $G(A)$ has at least one cycle with nonzero signed length. Assume that the cyclic index of $A$ is greater than 1. Then $W(A)$ cannot be a circular disk, and moreover if $W(A)$ is a regular polygon then its center must be at the origin.*

*Proof.* Let $m$ $(> 1)$ be the cyclic index of $A$. Assume first that $W(A)$ is a circular disk. In view of Theorem 3.12, each of the $m$ points $w(A)e^{2\pi ti/m}$, $t = 0, 1, \ldots, m-1$, is an extreme point of $W(A)$. Certainly, all of them lie on the circumference of the circular disk $W(A)$, and the center of the disk must be equidistant from all of them. It follows that the center of the disk is the origin of the complex plane. In other words, $W(A)$ is the circular disk with center at the origin and radius $w(A)$, in contradiction with the result of Theorem 3.12.

The same argument also shows that if $W(A)$ is a regular polygon, then its center must be at the origin. □

**Corollary 4.9.** *If $A$ is an irreducible nonnegative matrix with index of imprimitivity greater than 1, then $W(A)$ cannot be a circular disk.*

It seems plausible that this is the case for any irreducible nonnegative matrix $A$. Here we verify it for 2-by-2 matrices.

**Proposition 4.10.** *No irreducible nonnegative 2-by-2 matrix can have a circular disk as its numerical range.*

*Proof.* Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 irreducible nonnegative matrix whose numerical range is a circular disk with center $\lambda$ and radius $r$. It is known that in this case both eigenvalues of $A$ are equal to $\lambda$ and $A$ is unitarily similar to $\begin{bmatrix} \lambda & 2r \\ 0 & \lambda \end{bmatrix}$. We have $a + d = 2\lambda$ and $ad - bc = \lambda^2$. It follows that $ad - bc = (a + d)^2/4$, and thus $0 \leq (a - d)^2 = -4bc \leq 0$. This shows that $b = 0$ or $c = 0$. In either case, $A$ is reducible contradicting our assumption. □

An alternative way to complete the proof of Proposition 4.10 is to apply the Perron-Frobenius theory: since $A$ is unitarily similar to $\begin{bmatrix} \lambda & 2r \\ 0 & \lambda \end{bmatrix}$, we have, $\lambda$ equals $\rho(A)$ and is not a simple eigenvalue, contradiction.

The preceding argument can also be used to show that if $A$ is a 3-by-3 or 4-by-4 primitive matrix with zero trace, then $W(A)$ cannot be a circular disk. For if $W(A)$ is a circular disk with center $\lambda$, then by [CT, Remark 2] $\lambda$ is a (real) eigenvalue of $A$ with multiplicity at least two, hence $\lambda < \rho(A)$. On the other hand, by Corollary 3.4 and also the fact that $\rho(A)$ is the only eigenvalue of $A$ with modulus $\rho(A)$, $\lambda$ must be a positive number. If $A$ is 3-by-3, then tr $A = \rho(A) + 2\lambda > 0$, contradiction. If $A$ is 4-by-4, then the eigenvalues of $A$ are $\rho(A)$, $\lambda$, $\lambda$, and $-(\rho(A) + 2\lambda)$, which is again a contradiction, as $|-(\rho(A) + 2\lambda)| > \rho(A)$.

In [NT, Example 4.5] Nylen and Tam gave an example of an irreducible doubly stochastic matrix with index of imprimitivity two for which $W(A)$ is not a line segment (2-polygon). In related to that, we make the following simple observation:

**Remark 4.11.** Let $A$ be a real matrix. Then $W(A)$ is a line segment if and only if either $A$ is symmetric or $A$ is the sum of a real scalar matrix and a skew-symmetric matrix. If $A$ is nonnegative, then $W(A)$ is a line segment if only if $A$ is symmetric.

Indeed, first note that $W(A)$ is symmetric with respect to the real axis for the real $A$. Hence if $W(A)$ is a line segment, it will either be lying in $\mathbb{R}$ or be perpendicular to $\mathbb{R}$. In the former case, $A$ is symmetric. For the latter, assuming that $W(A)$ lies in the vertical line $x = a$, we have $W(i(A - aI)) \subseteq \mathbb{R}$. Thus $A = aI + B$ with $B = -i(i(A - aI))$ skew-symmetric. The converse is trivial. If $A$ is nonnegative and $W(A)$ is a line segment, then $W(A)$ cannot be perpendicular to the real axis for otherwise $w(A)$ would not be in $W(A)$. In this case, $W(A) \subseteq \mathbb{R}$ and hence $A$ is symmetric.

The more general question of when the numerical range of an irreducible nonnegative matrix is a regular polygon with center at the origin is actually already answered by Tam and Yang [TY, Remark 15]:

**Remark 4.12.** For any irreducible nonnegative matrix $A$ with index of imprimitivity $m \geq 2$, $W(A)$ is a regular polygon (necessarily with center at the origin) if and only if $W(A) = \operatorname{conv}\{\rho(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\}$.

Remark 4.12 also settles Issos's question, mentioned at the beginning of this section, for almost all cases except for the primitive matrix case. Our argument can also be used to show that, if $A$ is a primitive matrix, then $W(A)$ can never be a regular polygon with center at the origin. Certainly, there are primitive matrices whose numerical ranges are regular polygons. For instance, take an irreducible nonnegative matrix $A$ with index of imprimitivity $m > 1$ such that $W(A)$ is a regular polygon with center at the origin. Then for any $\alpha > 0$, $A + \alpha I$ is a primitive matrix whose numerical range is a regular polygon (with center at $\alpha$). The problem of characterizing primitive matrices with regular polygons as their numerical ranges remains open.

Contrary to what is said in [TY, p. 218, first paragraph], the condition given in Remark 4.12 can be transformed to a checkable condition. First, we observe the following:

**Lemma 4.13.** *Let $A \in M_n$. Let $\rho$ be a positive real number and let $m \geq 2$ be a given positive integer. In order that*

$$W(A) \subseteq \operatorname{conv}\{\rho e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\},$$

*it is necessary and sufficient that for $t = 0, 1, \ldots, m-1$, we have*

$$\lambda_{\max}(\operatorname{Re}(e^{-(2t-1)\pi i/m}A)) \leq \rho \cos\frac{\pi}{m}.$$

*Proof.* The polygon $\operatorname{conv}\{\rho e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\}$ can be expressed as $\bigcap_{t=0}^{m-1} H_t$, where $H_t$ is the closed half-plane given by:

$$H_t = \left\{ z = x + iy \in \mathbb{C} : x\cos\frac{(2t-1)\pi}{m} + y\sin\frac{(2t-1)\pi}{m} \leq \rho\cos\frac{\pi}{m} \right\}.$$

In order that $W(A)$ be included in the said polygon, it is necessary and sufficient that $W(A) \subseteq H_t$ for all $t$. Now $W(A) \subseteq H_t$ if and only if $W(e^{-(2t-1)\pi i/m}A)$ is included in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z \leq \rho\cos(\pi/m)\}$, and the latter condition is fulfilled if and only if $\lambda_{\max}(\operatorname{Re}(e^{-(2t-1)\pi i/m}A)) \leq \rho\cos(\pi/m)$. So our assertion follows. □

Now we have the following:

**Proposition 4.14.** *Let $A$ be an irreducible nonnegative matrix with index of imprimitivity $m$. In order that $W(A)$ be a regular polygon with center at the origin it is necessary and sufficient that the following conditions are both satisfied:*

(a) $\rho(A) = \rho(\mathrm{Re}(A))$.

(b) *For* $t = 0, 1, \ldots, m-1$, $\lambda_{\max}(\mathrm{Re}(e^{-(2t-1)\pi i/m}A)) = \rho(A)\cos(\pi/m)$.

(*In condition* (b), *we may replace the last equality by* "$\leq$".)

*Proof.* "Necessity": Suppose that $W(A)$ is a regular polygon with center at the origin. Then $w(A)$ must be a sharp point of $W(A)$ and, as noted before, condition (a) necessarily holds. Furthermore, by Remark 4.12, in this case we have

$$W(A) = \mathrm{conv}\,\{\rho(A)e^{2\pi ti/m} : t = 0, 1, \ldots, m-1\}.$$

It follows from Lemma 4.13 that for $t = 0, 1, \ldots, m-1$, we have

$$\lambda_{\max}(\mathrm{Re}(e^{-(2t-1)\pi i/m}A)) \leq \rho(A)\cos\frac{\pi}{m}.$$

Here we can replace each of the latter inequalities by an equality, because $W(A)$ is precisely the convex hull of the $m$ points $\rho(A)e^{2\pi ti/m}$, $t = 0, 1, \ldots, m-1$, not just a subset of it. So we have condition (b).

"Sufficiency": Again by Lemma 4.13, condition (b) implies that $W(A)$ is included in the regular polygon with vertices $\rho(A)e^{2\pi ti/m}$, $t = 0, 1, \ldots, m-1$. These are the same as $w(A)e^{2\pi ti/m}$ by condition (a). But by Theorem A, $W(A)$ also contains each of these $m$ points. Hence, $W(A)$ is equal to the said regular polygon. □

In view of Corollary 4.4, Theorem 4.7 and Proposition 4.14, in theory (assuming that all numerical quantities can be computed exactly), we can determine whether the numerical range of a nonnegative matrix is a regular polygon with center at the origin in the following way:

By a permutation similarity, we may rewrite the given nonnegative matrix $A$ as $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are each nonnegative matrices with connected undirected graph. Then we follow the steps given below. If we obtain positive answers at each step, then $W(A)$ is a regular polygon with center at the origin. Otherwise, $W(A)$ is not.

Step 1. For each $j = 1, \ldots, k$, determine the values of $\rho(A_j)$ and $w(A_j)$ ($= \rho(\mathrm{Re}\,A_j)$). (Then $\rho(A) = \max_{1 \leq j \leq k} \rho(A_j)$ and $w(A) = \max_{1 \leq j \leq k} w(A_j)$.) Answer the following question: Is there a $j$ such that $\rho(A_j) = \rho(A)$ and $A_j$ satisfies the criterion for $W(A_j)$ to be a regular polygon with center at the origin, as given by Proposition 4.14?

Step 2. Let $\Lambda$ denote the set of all $j$ for which $\rho(A_j) = \rho(A)$ and $W(A_j)$ is a regular polygon with center at the origin. For each $j \in \Lambda$, determine the index of imprimitivity $m_j$ of $A_j$ (for instance, by finding the greatest common divisor of the circuit lengths of $G(A_j)$). Answer the following question: Is there a $j_0 \in \Lambda$ such that $m_j$ divides $m_{j_0}$ for each $j \in \Lambda$? (If such $j_0$ exists, hopefully $W(A)$ equals $W(A_{j_0})$.)

Step 3. Answer the following question: Is it true that, for each $j$ for which $\rho(A_j) < \rho(A)$ or $\rho(A_j) = \rho(A)$ but $j \notin \Lambda$, we have $W(A_j) \subseteq W(A_{j_0})$? (Use Lemma 4.13 here.)

If we expect that $W(A)$ is not a regular polygon with center at the origin, we may also add the following step at the beginning:

Step 0. For $j = 1, \ldots, k$, determine $\rho(A_j)$ and $w(A_j)$ $(= \rho(\operatorname{Re} A_j))$. Answer the following questions:

(i) Is $\rho(A) = w(A)$?

(ii) For each $j$ for which $\rho(A_j) = \rho(A)$, is $A_j$ an irreducible matrix and do we have $\rho(A_j) = w(A_j)$?

(iii) For each $j$ for which $\rho(A_j) < \rho(A)$, do we have $w(A_j) < \rho(A)$?

(If the answers are all "yes", then $w(A)$ is a sharp point of $W(A)$.)

Now we would also like to address the question of when the numerical range of a nonnegative matrix $A$ has weak circular symmetry, i.e., $e^{2\pi i/m}W(A) = W(A)$ for some integer $m$, $2 \leq m \leq n$, where $n$ is the size of $A$. The question was solved for the special case when the undirected graph of $A$ is connected (see [TY, Theorem 2]). Clearly, the convex sets $W(A)$ and $W(e^{2\pi i/m}A)$ are equal if and only if they have same supporting lines in all directions. So one may give the following answer to the above question:

$$e^{2\pi i/m}W(A) = W(A) \quad \textit{if and only if}$$
$$\lambda_{\max}(\operatorname{Re}(e^{i\theta}A)) = \lambda_{\max}(\operatorname{Re}(e^{i(\theta+2\pi/m)}A) \quad \forall \theta \in [0, 2\pi).$$

But this is not a satisfactory answer, as there are infinitely many conditions we need to check. One may also try to reduce the problem to the case of a nonnegative matrix with connected undirected graph, and suspect that a result similar to Theorem 4.7 or [TY, Theorem 3] also holds for the question of weak circular symmetry. The following example shows that this is not the case.

**Example 4.15.** Choose an irreducible nonnegative matrix $A_1$ whose numerical range is the triangle $\triangle = \operatorname{conv}\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. Also choose a nonnegative block-shift matrix $A_2$ with connected undirected graph such that the numerical range of $A_2$ is a circular disk, centered at origin, radius $r$, where $r$ is greater than the radius of the inscribed circle of $\triangle$ but is less than that of the circumscribed circle. Now let $A = A_1 \oplus A_2$. Then $e^{2\pi i/3}W(A) = W(A)$, but we have neither $W(A_1) \subseteq W(A_2)$ nor $W(A_2) \subseteq W(A_1)$.

Note that in the above example $W(A)$ is not a polygon. But by modifying the example, we can easily construct one in which $W(A)$ is a (nonregular) polygon. The method of construction of our examples also suggests the following question:

**Question 4.16.** *Let $A$ be a nonnegative matrix which is permutationally similar to $A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are nonnegative matrices each with a connected undirected graph. If, for some positive integer $m \geq 2$, we have $e^{2\pi i/m}W(A) = W(A)$, does it follow that there exist distinct indices $i_1, \ldots, i_p \in \langle k \rangle$, $p \geq 1$, such that $e^{2\pi i/m}W(A_{i_r}) = W(A_{i_r})$ for $r = 1, \ldots, p$, and $W(A_j) \subseteq \operatorname{conv}(W(A_{i_1}) \cup \cdots \cup W(A_{i_p}))$ for all $j \neq i_1, \ldots, i_p$?*

We do not know the answer to the above question even when $W(A)$ is assumed to be a polygon. Also, note that if we drop the nonnegativity of $A$, the answer to the above question is clearly "no".

## Acknowledgement

This research started when the three authors were attending the Fifth Workshop on Numerical Ranges and Numerical Radii, Nafplio, Greece, June, 2000. We have learned that J. Maroulas, P.J. Psarrakos and M.J. Tsatsomeros had shown in their paper "Perron-Frobenius type results on the numerical range" that no irreducible nonnegative matrix can have a circular disk or an elliptic disk with foci off the real axis as its numerical range. This answers affirmatively the question we ask, preceding Proposition 4.10. We thank them for sending us their preprint. Thanks are also due to John Drew for a stimulating discussion with the first author that led to Example 3.5.

## References


[BP] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Revised reprint of the 1979 original, Classics in Applied Mathematics 9, SIAM, Philadelphia, 1994.

[CT] M.T. Chien and B.S. Tam, Circularity of the numerical range, *Linear Algebra Appl.* **201** (1994), 113–133.

[DW] A. Dritschel and H.J. Woerdeman, Model theory and linear extreme points in the numerical radius unit ball, *Memoirs Amer. Math. Soc.* **129** (1997), no.615.

[E] M.R. Embry, The numerical range of an operator, *Pacific J. Math.* **32** (1970), 647–650.

[F] M. Fiedler, Geometry of the numerical range of matrices, *Linear Algebra Appl.* **37** (1981), 81–96.

[GR] K.E. Gustafson and D.K.M. Rao, *Numerical Range: the Field of Values of Linear Operators and Matrices*, Springer, New York, 1996.

[GT] M. Goldberg and E. Tadmer, On the numerical radius and its applications, *Linear Algebra Appl.* **42** (1982), 263–284.

[HJ1] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[HJ2] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[I] J.N. Issos, The field of values of non-negative irreducible matrices, Ph.D. thesis, Auburn University, 1966.

[M] H. Minc, *Nonnegative Matrices*, Wiley, New York, 1988.

[NT] P. Nylen and T.Y. Tam, Numerical range of a doubly stochastic matrix, *Linear Algebra Appl.* **153** (1991), 161–176.

[P] V. Pták, Isometric parts of an operator and the critical exponent, *Časopis Pěst. Mat.* **101** (1976), 383–388.

[T1] B.S. Tam, Circularity of numerical ranges and block-shift matrices, *Linear and Multilinear Algebra* **37** (1994), 93–109.

[T2] B.S. Tam, On matrices with cyclic structure, *Linear Algebra Appl.* **302–303** (1999), 377–410.

[TY] B.S. Tam and S. Yang, On matrices whose numerical ranges have circular or weak circular symmetry, *Linear Algebra Appl.* **302–303** (1999), 193–221.

[W] H. Wielandt, Unzerlegbare, nicht negative Matrizen, *Math. Z.* **52** (1950), 642–648.

# 出席國際會議報告

會議名稱：第十屆國際線性代數學會會議

會議地點：美國奧本市奧本大學

會議時間：91 年 6 月 10 日至 13 日

報 告 人：淡江大學數學系　譚必信

撰寫日期：91 年 6 月 26 日

《國際線性代數學會會議》每三年舉辦兩次，今年是第十屆，輪到在美國奧本大學舉行，參加者來自世界各地，共約 130 多人。

在兩個全天及兩個半天的緊湊議程中總共安排了 127 場演講，其中包括 8 場一小時的大會演講、15 個半小時的邀請演講、33 個半小時的迷你會議演講及 71 個 20 分鐘的分組報告。這次會議總共安排了 6 個迷你會議，主題包括：Complexity in numerical linear algebra、Matrices in control problems、Matrix extensions and interpolation problems、Linear algebra education、Matrices in max algebras 及 Nonlinear matrix equations。分組報告也有分主題，包括：Optimization、Eigenvalues、General matrices、Inverse eigenvalue problems、Special matrices、Rank、Graphs、Control、Majorization、Preservers、Inverses、Positive matrices、Matrix function 及 Sign patterns 等等。

這屆的 Hans Schneider Prize 是由日本的 T. Ando 及加拿大的

P. Lancaster 教授分別獲得，而這次會議的 Hans Schneider Prize 演講則是由 T. Ando 主講，講題爲“Cones and norms in the tenser-product space”。

除了一小時的大會報告以外，會議的演講都是平行進行。本人主要選擇聆聽主題跟自己興趣相關的幾個分組報告，其中包括：Majorization、Preservers、Positive Matrices 等等。在這次會議我聽到的精彩（或感興趣的）演講有 16 個以上。特別欣賞的爲蔡文端教授的大會演講“The norm estimate for the sum of two matrices”（是與李志光教授合作），真是深入淺出，令人有無限的想像空間。另外，Naomi Shaked-Monderer 的“Minimal cp-rank”及 F. Barioli 的“Complete positivity of small and large actue set of vectors”都與目前本人與以色列 R. Loewy 教授合作的完全正矩陣的研究有密切關連。

在會議的第一天上午本人應邀擔任分組“General Matrices”的主持人。本人的演講是安排在會議第三天早上的分組“Preservers”，講題爲“Strong linear preservers of symmetric doubly stochastic matrices”，是報告與學生林淑慧女士合作的研究工作。我的演講是引起一些注意。

攜回資料：會議議程及摘要一本。

# Strong Linear Preservers of Symmetric Doubly Stochastic Matrices

by

Bit-Shun Tam
Tamkang University

(This is a joint work with Shwu-Huey Lin.)

$T : V \to V$ a linear mapping, $S \subseteq V$ s.t. span $S = V$.

$T$ *preserves* (resp., *strongly preserves*) $S$ if $T(S) \subseteq S$ (resp., $T(S) = S$).

$\mathbf{P}(n)$ : the group of $n \times n$ permutation matrices;

$\mathbf{A}(n)$ : the group of $n \times n$ even permutation matrices;

$\mathbf{DS}(n)$ : the set of $n \times n$ doubly stochastic matrices;

$\mathbf{SDS}(n)$ : the set of $n \times n$ symmetric doubly stochastic matrices.

$\mathcal{E}(C)$ : the set of extreme points of the convex set $C$.

**Theorem A (Li, Tam & Tsing).** *Let $T$ be a linear map on* $\text{span}(\mathbf{DS}(n))$. *T.F.A.E.:*

(a) $T(\mathbf{DS}(n)) = \mathbf{DS}(n)$.

(b) $T(\mathbf{P}(n)) = \mathbf{P}(n)$.

(c) *$T$ is given by $T(X) = PXQ$ or $T(X) = PX^tQ$ for some $P, Q \in \mathbf{P}(n)$.*

**Lemma A (Chiang & Li).** *For* $n \geq 4$, $\text{span } \mathbf{A}(n) = \text{span } \mathbf{DS}(n) =$ *the space of* $n \times n$ *real matrices with equal row and column sums.*

**Theorem B (Chiang & Li).**

(i) $\mathbf{A}(2) = \{I_2\}$; *the identity operator is the only strong linear preserver.*

(ii) $\mathbf{A}(3)$ *consists of* 3 *lin. indep. matrices*; *so* $T(\mathbf{A}(3)) = \mathbf{A}(3)$ *iff* $T$ *permutes the elements of* $\mathbf{A}(3)$.

(iii)
$$\begin{aligned} H_0 &:= \{P \in \mathbf{A}(4) : P^2 = I_n\} \\ &= \{I_4, \mathbf{P}((1,2,)(3,4)), \mathbf{P}((1,3,)(2,4)), \mathbf{P}((1,4)(2,3))\}; \end{aligned}$$

$H_1$, $H_2$, *cosets of* $H_0$ *in* $\mathbf{A}(4)$;

*If* $T$ *is a strong linear preserver of* $\mathbf{A}(4)$, *then*

$$T(H_j) = H_{i_j} \text{ for } j = 0, 1, 2 \text{ with } \{i_0, i_1, i_2\} = \{0, 1, 2\}. \qquad (*)$$

*Conversely, if* $\psi : \mathbf{A}(4) \to \mathbf{A}(4)$ *is a bijection s.t.* $(*)$ *holds, then* $\psi$ *can be extended uniquely to a linear map on span* $\mathbf{A}(n)$.

(iv) *Let* $n \geq 5$. *A linear map* $T : \text{span } \mathbf{A}(n) \to \text{span } \mathbf{A}(n)$ *satisfies* $T(\mathbf{A}(n)) = \mathbf{A}(n)$ *iff* $\exists\, P,\ Q \in \mathbf{P}(n)$ *with* $PQ \in \mathbf{A}(n)$ *s.t.* $T$ *is given by:*

$$T(X) = PXQ \text{ or } T(X) = PX^tQ.$$

**Theorem 1 (Lin & Tam).** *Let* $T$ *be a linear map on* span $\mathbf{SDS}(n)$, $n \geq 3$. *T.F.A.E.:*

(a) $T(\mathbf{SDS}(n)) = \mathbf{SDS}(n)$.

(b) $\exists\, P \in \mathbf{P}(n)$ *s.t.* $T(X) = P^tXP \quad \forall X$.

The implication (b) $\Longrightarrow$ (a) is obvious.

Will sketch the proof of (a) $\Longrightarrow$ (b).

Hereafter, we assume that $T$ is a strong linear preserver of $\mathbf{SDS}(n)$.

We need the following characterization:

**Theorem (M. Katz, 1970):** *The extreme points of* $\mathbf{SDS}(n)$ *are those matrices which are permutationally similar to direct sum of (some of) the following three types of matrices:*

(i) $[1]$, $1 \times 1$ *matrix,*

(ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $2 \times 2$ *matrix,*

(iii) $$\begin{bmatrix} 0 & 1/2 & & & 1/2 \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & \ddots & \\ & & \ddots & \ddots & 1/2 \\ 1/2 & & & 1/2 & 0 \end{bmatrix}_{k \times k}, \quad k \geq 3, \textit{ odd.}$$

**Lemma 1.** $C_2$ := *the collection of all transposition matrices of* $\mathbf{P}(n)$. *Then* $C_2 \cup \{I_n\}$ *is a basis for* span $\mathbf{SDS}(n)$.

**Assertion 1.** $T(I_n) = T(I_n)$.

A difficult part of the proof, will come back later.

$$J_n := \text{ the } n \times n \text{ matrix with all entries equal } 1$$
$$\widetilde{J}_n := J_n - I_n$$

**Assertion 2.** $T(\widetilde{J}_n) = \widetilde{J}_n$.

$T$ permutes the elements of $\mathcal{E}(\mathbf{SDS}(n))$.

$$T\left(\sum_{P \in \mathcal{E}(\mathbf{SDS}(n))} P\right) = \sum T(P) = \sum P.$$

By symmetry, $\sum P = \beta_n I_n + \gamma_n \widetilde{J}_n$, where $\beta_n,\ \gamma_n > 0$. So $T$ fixes $\beta_n I_n + \gamma_n \widetilde{J}_n$. But $T$ also fixes $I_n$, so $T$ fixes $\widetilde{J}_n$.

**Assertion 3.** $T(\mathcal{C}_2) = \mathcal{C}_2$.

$T$ fixes $\sum_{A \in \mathcal{C}_2} A$, as the latter equals $\widetilde{J}_n + \frac{(n-1)(n-2)}{2} I_n$; so

$$\begin{aligned}(n-2)|\mathcal{C}_2| &= \sum_{A \in \mathcal{C}_2} \operatorname{tr} A \\ &= \sum_{A \in \mathcal{C}_2} \operatorname{tr} T(A) \\ &\geq (n-2)|\mathcal{C}_2|,\end{aligned}$$

hence $T(A) \in \mathcal{C}_2 \ \ \forall\ A \in \mathcal{C}_2$.

**Assertion 4.** *Suppose that $T(P(i,j)) = P(p,q)$ and $T(P(k,l)) = P(r,s)$. If $\{i,j\} \cap \{k,l\}$ is a singleton, then so is $\{p,q\} \cap \{r,s\}$.*

**Assertion 5.** $\exists P \in \mathbf{P}(n)$ *s.t.* $T(X) = P^t X P \quad \forall X \in \mathcal{C}_2$.

In view of Lemma 1 and Assertion 5, we have (a) $\Longrightarrow$ (b).

Now back to the proof of Assertion 1, that $T(I_n) = I_n$.

In [Li–Tam–Tsing], by replacing $T$ by $\widetilde{T}$ defined by $\widetilde{T}(X) = T(I_n)^t T(X)$, it is assumed that a strong linear preserver $T$ of $\mathbf{DS}(n)$ fixes $I_n$.

Here we exploit the concept of neighborly extreme points of a polytope.

Two extreme points $x$, $y$ of a polytope $C$ are ***neighborly*** if the line segment joining $x$, $y$ is a face of $C$, or equivalently, the face $\Phi(\frac{x+y}{2})$ contains precisely two extreme points, namely, $x$ and $y$.

$$N(x) := \{y \in \mathcal{E}(C) : y \text{ is neighborly to } x\} \quad (x \in \mathcal{E}(C)).$$

Let $T$ be a strong linear preserver of $C$. Then $TN(x) = N(Tx) \ \ \forall x \in \mathcal{E}(C)$, $T$ maps $\mathcal{E}_k := \{x \in \mathcal{E}(C) : |N(x)| = k\}$ onto itself, for any $k \in \mathbb{Z}_+$, and

$\forall x \in \mathcal{E}(C), |\mathcal{E}_k \cap N(x)| = |\mathcal{E}_k \cap N(Tx)| \;\; \forall k \in \mathbb{Z}_+.$

By the ***graph*** of an $n \times n$ symmetric matrix $A$, denoted by $G(A)$, we mean the graph with vertex set $\langle n \rangle := \{1, \dots, n\}$, where $(i, j)$ is an arc iff $a_{ij} \neq 0$.

We call a graph an *LOCC* graph if its connected components are each either a line segment or an odd cycle (including a loop).

By Katz's theorem, for any $A \in \mathcal{E}(\mathbf{SDS}(n))$, $G(A)$ is an LOCC graph.

Two LOCC graphs $G$, $H$ on the same vertex set are ***neighborly*** if their union $G \cup H$ contains $G$ and $H$ as its only spanning LOCC subgraphs.

For any $A,\ B \in \mathcal{E}(\mathbf{SDS}(n))$, $A$ and $B$ are neighborly iff the LOCC graphs $G(A)$ and $G(B)$ are neighborly.

$N(G) :=$ collection of LOCC graphs neighborly to $G$.

$G_i^n :=$ collection of LOCC graphs on $\langle n \rangle$ having precisely $i$ line segments among their connected components.

**Theorem 2.** *Given $n \in \mathbb{Z}_+$, $n \geq 3$. For $i = 0, \ldots, [n/2]$, $|N(G)|$ is independent of the choice of $G$ from $G_i^n$. If $N_i^n$ denotes the common value of $|N(G)|$ for $G \in G_i^n$, then*

$$N_0^n < N_1^n < \cdots < N_{[n/2]}^n.$$

Assuming Theorem 2, we have

**Proof of Assertion 1.** For each $i = 0, \ldots, [n/2]$,

$$\begin{aligned} E_i^n &:= \{A \in \mathcal{E}(\mathbf{SDS}(n)) : G(A) \in G_i^n\} \\ &= \{A \in \mathcal{E}(\mathbf{SDS}(n)) : |N(A)| = N_i^n\} \end{aligned}$$

So $T(E_i^n) = E_i^n$.

For even $n$:

We have $\sum_{A\in E^n_{[n/2]}} A = \alpha_n \widetilde{J}_n$ for some $\alpha_n > 0$, so $T$ fixes $\widetilde{J}_n$. But $T$ also fixes $\sum_{A\in \mathcal{E}(SDS(n))} A$, which is of the form $\beta_n I_n + \gamma_n \widetilde{J}_n$ with $\beta_n,\ \gamma_n > 0$.

So $T$ fixes $I_n$.

For odd $n$, $n \geq 5$ ($n = 3$ can be treated separately):

We have $\sum_{A\in E^n_{[n/2]}} A = \alpha_n J_n$ for some $\alpha_n > 0$, so $T$ fixes $J_n$.

We have $E^n_{[n/2]-1} = \mathcal{L} \,\dot\cup\, \mathcal{T}$,

$$\text{where } \mathcal{L} := \{A \in E^n_{[n/2]-1} : G(A) \text{ contains three loops}\}$$
$$\mathcal{T} := \{A \in E^n_{[n/2]-1} : G(A) \text{ contains a 3-cycle}\}.$$

Using a technical lemma on LOCC graphs, we show that $T$ preserves the sets $\mathcal{L}$ and $\mathcal{T}$. So $T$ fixes $\sum_{A\in\mathcal{T}} A$, which is of the form $\omega \widetilde{J}_n$, where $\omega > 0$. But we already have, $T$ fixes $J_n$, so $T$ fixes $I_n$.

The proof of Theorem 2 relies on the following:

**Theorem 3.** *A connected graph is the union of two neighborly LOCC graphs iff it is one of the following:*

(a) *a path of length* $\geq 1$ *with odd cycles attached at its two ends (such that the path and the two odd cycles are pairwise internally disjoint);*

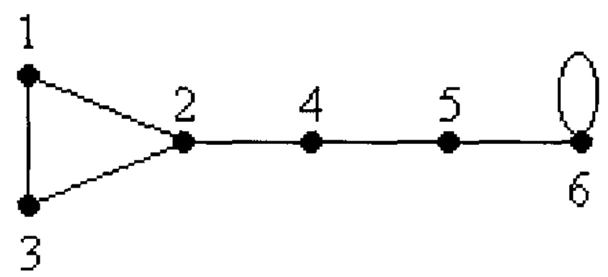
(b) *an odd cycle of length* $\geq 3$ *with an odd path (open or closed, internally disjoint from the cycle) joining two (not necessarily distinct) vertices of the cycle; or*

(c) *an even cycle of length* $\geq 4$.

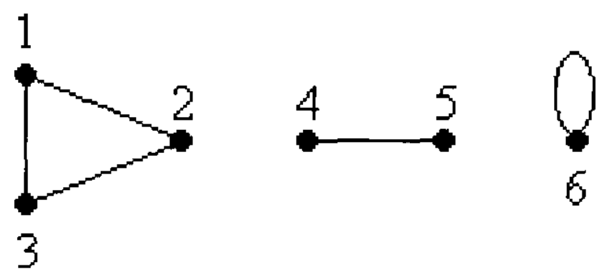
The proof of the "if" part is more straightforward.

For instance,

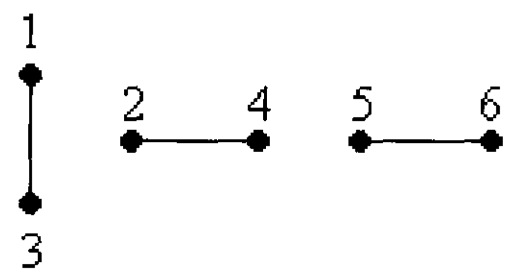
(a)



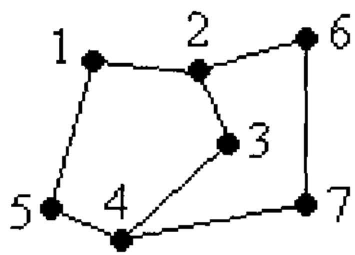

is the union of the neighborly LOCC graphs



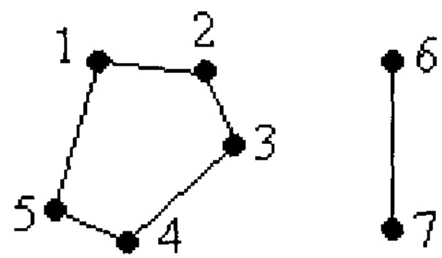

and



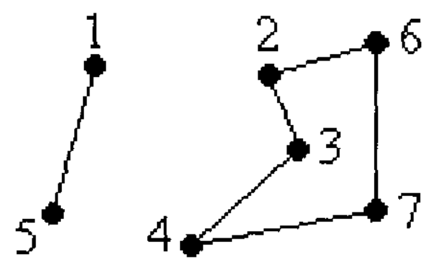

(b)



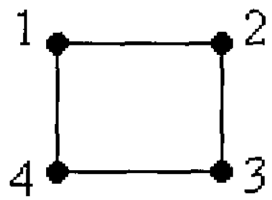
is the union of the neighborly LOCC graphs



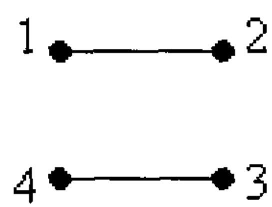

and



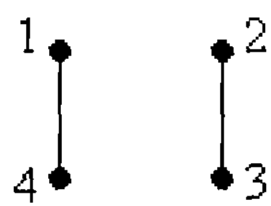

(c)




is the union of the neighborly LOCC graphs




and




The proof of the "only if" part takes 5 single-spaced typed pages.

A crucial step in the proof of Theorem 1, (a) $\Longrightarrow$ (b) is Assertion 1 (i.e., $T(I) = I$). Originally, we tried to prove it without using the concept of LOCC graphs. We succeeded in reducing the problem to proving the following conjecture:

**Conjecture.** Let $h(n)$ denote $|\mathcal{E}(\mathbf{SDS}(n))|$. For each positive integer $n \geq 3$, $(n-1)h(n-1)$ is not divisible by $h(n) - h(n-1)$.

We know we have the following recurrence relation, which holds for all $n \geq 4$:

$$\begin{aligned} h(n) = h(n-1) + (n-1)^2 h(n-2) - \frac{1}{2}(n-1)(n-2)h(n-3) \\ -(n-1)(n-2)(n-3)h(n-4) \end{aligned}$$

where $h(0) := 1$. But we cannot solve the recurrence relation. By running a computer, we verify the conjecture for $n = 3, \ldots, 171$. But for $n \geq 172$, the numbers involved are too large (larger than $10^{305}$) and data overflow.

A related result:

**Theorem 4 (Lin & Tam).** *Let $T$ be a linear map on the space of $n \times n$ real symmetric matrices, $n \geq 1$. Let $\mathbf{SDsS}(n)$ denote the polytope of all $n \times n$ real symmetric doubly substochastic matrices. T.F.A.E.:*

(a) $T(\mathbf{SDsS}(n)) = \mathbf{SDsS}(n)$.

(b) $\exists\, P \in \mathbf{P}(n)$ *s.t.* $T(X) = P^t X P \quad \forall X$.

# References

1. H. Chiang and C.K. Li, Linear maps leaving the alternating group invariant, *Linear Algebra Appl.* **340** (2002), 69–80.

2. C.K. Li, B.S. Tam and N.K. Tsing, Linear maps preserving permutation and stochastic matrices, *Linear Algebra Appl.* **341** (2002), 5–22.