# 行政院國家科學委員會補助專題研究計畫成果報告

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※ 阿爾廷質根問題 ※

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計畫類別：☑個別型計畫　□整合型計畫

計畫編號：NSC 90－2115－M－032－003

執行期間： 90 年 08 月 01 日 至 91 年 07 月 31 日

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本成果報告包括以下應繳交之附件：

□赴國外出差或研習心得報告一份

□赴大陸地區出差或研習心得報告一份

□出席國際學術會議心得報告及發表之論文各一份

□國際合作研究計畫國外研究報告書一份

執行單位： 淡 江 大 學 數 學 系

中 華 民 國 91 年 10 月 10 日

一、中文摘要：

我們發現定義在有理數的一維的環面$\boldsymbol{Q}$. Let $\boldsymbol{T}$ be a one-dimensional torus over $\boldsymbol{Q}$, and let $P$ be a nontorsion point on $\boldsymbol{T(Q)}$. Under Generalized Riemann Hypothesis, we derive an explicit density formula for the set of rational primes $p$ such that $P$ modulo $p$ generates $\boldsymbol{T(F_p)}$.

關鍵詞：質根, 環面, 密度。

二、英文摘要(**Abstract**):

We exploit an analogue of Artin's primitive roots conjecture for one dimension tori over $\boldsymbol{Q}$. Let $\boldsymbol{T}$ be a one-dimensional torus over $\boldsymbol{Q}$, and let $P$ be a nontorsion point on $\boldsymbol{T(Q)}$. Under Generalized Riemann Hypothesis, we derive an explicit density formula for the set of rational primes $p$ such that $P$ modulo $p$ generates $\boldsymbol{T(F_p)}$.

關鍵詞(Key Words): primitive root, torus, density.

三、計畫緣由與目的：

Let $\boldsymbol{T}$ be a one-dimensional torus over the rational numbers, and let $P=(x_0,y_0)$ be a nontorsion rational point. We are interested in the set $M_P$ consisting of rational primes $p$ where $\boldsymbol{T}$ has good reduction and $P$ modulo $p$ generates the abelian group $\boldsymbol{T(F_p)}$. The case $\boldsymbol{T}= \boldsymbol{G_m}$ dates back to Artin. The well-known Artin's conjecture (1927) asserts that for every nonzero non-square rational integer $a\neq\pm1$, the set of rational primes $p$ for which $a$ is a primitive root possesses a positive density. This conjecture was proved by Hooley[3] in 1967 under the Generalized Riemann Hypothesis (GRH). The purpose of this project is to generalize Hooley's Theorem to all one-dimensional tori over the rational numbers.

四、計畫結果與討論：

此一研究計畫中，我們主要得到以下的兩個定理：

定理一（Theorem 1）：

Let $\boldsymbol{T}$ be a one-dimensional torus over $\boldsymbol{Q}$, and let $P$ be a nontorsion point. Assume GRH holds. Then the set $M_P$ has a (Dirichlet) density, given by $\mathrm{den}(M_P)=\delta\cdot A$, where $A$ is the Artin's constant and $\delta$ is a rational number, which can be explicitly determined from $(\boldsymbol{T}, P)$. Moreover, we have $\delta$ if and only if $P$ is not a point in $\boldsymbol{T(Q)}^q$ for all primes $q$ dividing $\#\mathrm{Tor}(\boldsymbol{T(Q)})$.

定理二（Theorem 2）:

Let $K$ be a quadratic field and given $\alpha$ in element in $K$ with norm 1, which is not a root of unity. Assume GRH holds. Then den($M_\alpha$)= $\delta \cdot A$, where $\delta$ is a rational number explicitly determined from ($K$, $\alpha$). Moreover, we have $\delta>0$ if and only if (1) $\alpha$ is not a square when K is not equal to $\mathbf{Q}(\mu_3)$, or (2) $\alpha$ is neither a square nor a cube when K is equal to $\mathbf{Q}(\mu_3)$.

## 五、計畫成果自評

此一研究成果將有理數的Artin的質根猜想推廣至一維環面及二次數體的情況。其中需要大量地代數的理論支持以及相當繁瑣的計算，可是我們總算克服困難成功地得到上述的兩個定理，完成此一研究成果。

## 六、參考文獻


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