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ABSTRACT

We have completed the following works

1. Dirac electron in external fields [Phys. Rev. A61 (2000) 032104],

2. Mellin-transform in q-difference equation [Phys. Lett. A268 (2000) 217]

3. Symmetry breaking on torus [to appear in Phys. Rev. D]

4. Fractional fermion number [Submitted to Phys. Rev. D]

5. Quantum FK model [submitted to Phys. Rev. E]

6. Neutral fermion in external fields [Submitted to J. Phys. G].

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On the use of Mellin transform to a class of q-difference-differential equations

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Abstract

We explore the possibility of using the method of classical integral transforms to solve a class of q-difference-differential equations. The Laplace and the Mellin transform of q-derivatives are derived. The results show that the Mellin transform of the q-derivative resembles most closely the corresponding expression in classical analysis, and it could therefore be useful in solving certain q-difference equations. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. The study of q-analysis is an old subject, which dates back to the end of the 19th century [1–4]. It has found many applications in such areas as the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra, etc [5]. Recent developments in the theory of quantum group has boosted further interests in this old subject [6,7].

The subject of q-analysis concerns mainly the properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base, q. The relations among these functions, and the difference equations satisfied by them are among the topics most studied so far. The q-difference equations involve a new kind of difference operator, the q-derivative, which can be viewed as a sort of deformation of the ordinary derivative. Solutions of the q-difference equations in one variable have been well studied in terms of the q-hypergeometric series (also called the basic hypergeometric series). Partial q-difference equations and q-difference-differential equations with more than one variables are generally studied by means of the method of seperation of variables, or by the techniques of Lie symmetry in the literature [3,8–13]. The method of integral transforms, which is another powerful technique of solving differential equations in classical analysis, has not been, in our view, explored in q-analysis. The reason is not hard to understand. The main virtue of the classical integral transforms, particularly the Fourier and the Laplace transform, is to transform a differential equation into an algebraic equation, which can be solved easily. That this is possible is due to the fact that these transforms change the derivatives of a function to something proportional to the transform of the original function. As far as we know, integral transforms or q-integral transforms which could

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transform q-difference equations into algebraic equations have not been found. It should be mentioned that in fact q-analogues of Fourier transform, based on the Jackson q-integral, have been proposed recently [14,15]. However, in order for the q-Fourier transform of the q-derivative of a function $f(x)$ to be proportional to the q-Fourier transform of $f(x)$, the function $f(x)$ must satisfy very special conditions, such as $f(q^{-1})=0=f(-q^{-1})$ [15]. Hence, while these q-Fourier transforms may be useful in proving certain identities among the q-special functions, their use in solving q-difference equations seems limited.

In this paper we shall explore the possibility of using the method of classical integral transform to solve a class of q-difference-differential equations. We derive the Laplace and the Mellin transform of q-derivative, and argue that the Mellin transform, which is not generally employed in solving differential equations in classical analysis, may still be useful in solving certain q-difference equations.

2. Suppose we want to solve the following q-diffusion equation

$$D_t^q y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t) \quad (-\infty < x < \infty,\ t>0) \tag{1}$$

subject to the initial condition

$$y(x,0) = f(x). \tag{2}$$

Here D_t^q is the "forward" temporal q-derivative defined by [15,16]

$$D_t^q h(t) := \frac{h(q^{-1}t) - h(t)}{(1-q)t}. \tag{3}$$

for any function $h(x)$. We assume $0<q<1$ in this paper. The function $f(x)$ is assumed to vanish as $x \to \pm\infty$. One may as well use the more common definition of q-derivative [4]

$$\mathscr{D}_t^q h(t) := \frac{h(t) - h(qt)}{(1-q)t}. \tag{4}$$

We shall not employ this definition of the q-derivative here for reason to be explained later. We note here that q-difference and q-difference-differential equations of the diffusion type such as Eq. (1) have been considered before [9–13], but mostly from the point of view of Lie symmetry, or by seperation of variables.

We can remove the partial differential operator in x in (1) by a Fourier transform. The question now is to choose an appropriate integral transform to remove the q-derivative. In view of the positivity of the time variable, the two most natural choices are the Laplace and the Mellin transform.

Let us first derive the expression of the Laplace transform of the q-derivative. The Laplace transform of a function $h(t)$ is defined as $\bar{h}(s) := \mathscr{L}\{h(x),s\} = \int_0^\infty h(t)\exp(-st)dt$. For the q-derivative of $h(x)$, the Laplace transform is

$$\mathscr{L}\{D_t^q h(t),s\} = \frac{1}{1-q}\left[\int_0^\infty \frac{h(q^{-1}t)}{t} e^{-st}dt - \int_0^\infty \frac{h(t)}{t} e^{-st}dt\right]. \tag{5}$$

To proceed we have to use the following relation of the Laplace transform [18]

$$\int_s^\infty \bar{h}(s')\,ds' = \int_0^\infty \frac{h(t)}{t} e^{-st}dt, \tag{6}$$

provided the integral on the r.h.s. of (6) is well-defined. We may apply (6) to (5) directly if $h(0)=0$. However, if $h(0)\neq 0$, the r.h.s. of (6) is not well-defined, and direct application of (6) to (5) leads to incorrect result which does not reduce to the usual expression of the Laplace transform of derivative in the classical limit $q\to 1^-$. In order to recover the classical limit correctly, we find it necessary to regularise (5) in the form

$$\frac{1}{1-q}\left[\int_0^\infty \frac{h(q^{-1}t)-h(0)}{t} e^{-st}dt - \int_0^\infty \frac{h(t)-h(0)}{t} e^{-st}dt\right]. \tag{7}$$

We may now apply (6) to (7). Making use of

$$\mathscr{L}\{h(t)-h(0),s\} = \bar{h}(s) - s^{-1}h(0) \tag{8}$$

we finally obtained

$$\mathscr{L}\{D_t^q h(t),s\} = \frac{1}{1-q}\int_{sq}^{s} \bar{h}(s')\,ds' - \frac{\ln q^{-1}}{1-q}h(0). \tag{9}$$

Eq. (9) reduces to the expression $s\bar{h}(s) - h(0)$ for the Laplace transform of ordinary derivative as $q \to 1^-$.

If one uses instead the definition (4) for the q-derivative, the Laplace transform would be

$$\mathscr{L}\{\mathscr{D}_t^q h(t),s\} = \frac{1}{1-q}\int_{s}^{\frac{s}{q}} \bar{h}(s')\,ds' - \frac{\ln q^{-1}}{1-q}h(0). \tag{10}$$

It is now obvious that the Laplace transform is not useful in solving equations involving q-derivatives: it transforms such equations into integral equations!

3. We now consider the Mellin transform of a q-derivative. The Mellin transform is seldom being used in solving differential equations, because it generally transforms differential equations into difference equations instead of the much simpler algebraic equations. Now that the Fourier and the Laplace transform lose their virtues whenever q-derivatives are present, the Mellin transform is naturally the next one to be looked at. As we shall see below, the Mellin transform still transforms an equation containing q-derivatives into a difference equation of the transformed function, which is the best thing next to an algebraic equation one could get. Previously, the use of the Mellin transform in q-analysis is limited to proving various identities among the q-special functions [2,17].

The Mellin transform of a function $h(t)$ is defined as $h^*(s) := \mathscr{M}\{h(t),s\} = \int_0^\infty h(t)t^{s-1}dt$. For q-derivative defined in (3), we have

$$\mathscr{M}\{D_t^q h(t),s\} = -[s-1]_q\, h^*(s-1). \tag{11}$$

Here $[x]_q$ is the q-number defined by

$$[x]_q := \frac{1-q^x}{1-q}. \tag{12}$$

Note that $[x]_q \to x$ as $q \to 1^-$. Hence (11) reduces to the expression $-(s-1)h^*(s-1)$ for the Mellin transform of the ordinary derivative as $q \to 1^-$. Repeated use of (11) leads to

$$\begin{aligned}\mathscr{M}\{(D_t^q)^n h(t),s\} &= (-1)^n [s-1]_q [s-2]_q \cdots [s-n]_q \\ &\quad \times h^*(s-n), \quad n \geq 1. \end{aligned} \tag{13}$$

This is the q-analogue of the corresponding formula in the classical case [18].

For the definition (4), one has

$$\mathscr{M}\{\mathscr{D}_t^q h(t),s\} = [1-s]_q\, h^*(s-1) \tag{14}$$

$$= -q^{1-s}[s-1]_q\, h^*(s-1). \tag{15}$$

Here an extra factor of q appears compared with (11). In order to simplify our presentation, we therefore adopt the definition (3) in this paper. We must, however, mention that all the arguments given below apply equally well to the corresponding cases with q-derivatives replaced by the definition (4).

4. Let $Y^*(\xi,s)$ be the transformed function of $y(x,t)$ obtained by taking the Mellin transform in t and a Fourier transform

$$G(\xi) := \int_{-\infty}^{\infty} g(x)\exp(i\xi x)\,dx$$

in x. Making these transforms to (1), one obtains

$$[s-1]_q Y^*(\xi,s-1) = \xi^2 Y^*(\xi,s). \tag{16}$$

Fortunately solution to this equation can be readily found to be

$$Y^*(\xi,s) = A(\xi)\,\xi^{-2s}\Gamma_q(s), \tag{17}$$

where $A(\xi)$ is some function of ξ only, and $\Gamma_q(s)$ is the *q-gamma function* defined by [4]

$$\Gamma_q(s) := \frac{(q;q)_\infty}{(q^s;q)_\infty}(1-q)^{1-s},\ 0<q<1. \tag{18}$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k). \tag{19}$$

$\Gamma_q(s)$ satisfies

$$\lim_{q\to 1^-} \Gamma_q(s) = \Gamma(s)\,, \tag{20}$$

$$\Gamma_q(s+1) = [s]_q \Gamma(s)\,, \quad \Gamma_q(1) = 1\,. \tag{21}$$

Inverse-Mellin transform of $\xi^{-2s}\Gamma_q(s)$ in (17) is

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \xi^{-2s}\Gamma_q(s)t^{-s}ds\,. \tag{22}$$

The poles of $\Gamma_q(s)$ are $s = 0, -1, -2, \ldots$. The residual of $\Gamma_q(s)$ at pole $s = -n$ ($n \geq 0$) is [4]:

$$\frac{(1-q)^{n+1}}{(q^{-n};q)_n \ln q^{-1}}\,. \tag{23}$$

The symbol $(a;q)_n$ is the *q-shifted factorial*:

$$(a;q)_0 := 1\,, \quad n = 0\,, \tag{24}$$

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})\,,$$
$$n = 1,2\ldots \tag{25}$$

Hence (22) becomes

$$\frac{1-q}{\ln q^{-1}}\sum_{n=0}^{\infty}\frac{\left[(1-q)\xi^2 t\right]^n}{(q^{-n};q)_n}\,. \tag{26}$$

In view of the identity [4]

$$(q^{-n};q)_n = \left(-\frac{1}{q}\right)^n q^{-n(n-1)/2}(q;q)_n\,, \tag{27}$$

(26) can be expressed as

$$\frac{1-q}{\ln q^{-1}}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}\left[-q(1-q)\xi^2 t\right]^n$$
$$= \frac{1-q}{\ln q^{-1}} E_q\left(-q(1-q)\xi^2 t\right). \tag{28}$$

The function $E_q(z)$ (for complex z) is the *q-exponential function* defined by [4]

$$E_q(z) := \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}z^n}{(q;q)_n} = (-z;q)_\infty\,. \tag{29}$$

In the limit $q \to 1^-$, Eq. (28) tends to the usual exponential function $\exp(-\xi^2 t)$. Finally, performing an inverse Fourier transform we obtain the solution of the q-diffusion equation

$$y(x,t)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\xi)$$
$$\times\left\{\frac{1-q}{\ln q^{-1}} E_q\left(-q(1-q)\xi^2 x\right)\right\} e^{-i\xi x}\,d\xi\,. \tag{30}$$

Setting $t = 0$ in (30) shows that

$$\frac{1-q}{\ln q^{-1}} A(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,0)e^{i\xi x}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,dx$$
$$\equiv F(\xi) \tag{31}$$

is the Fourier transform of $y(x,0) = f(x)$. So the final solution of the initial problem is

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\xi)$$
$$\times E_q\left(-q(1-q)\xi^2 t\right)e^{-i\xi x}\,d\xi\,. \tag{32}$$

This is the q-analogue of the solution given in [19] for the corresponding classical case. One can easily check that (32) indeed satisfies (1) by using the following identity

$$D_t^q E_q(\lambda t) = \frac{\lambda}{q(1-q)} E_q(\lambda t)\,. \tag{33}$$

Let us consider an example. Suppose the initial profile is $f(x) = \exp(-x^2/4b)/\sqrt{2b}$, $(b > 0)$. Its Fourier transform is $F(\xi) = \exp(-b\xi^2)$. Then from (32) and (29), we get

$$y(x,t) = E_q\left(q(1-q)t\frac{d}{db}\right) f(x)\,. \tag{34}$$

In the limit $q \to 1^-$, Eq. (34) gives the classical solution

$$y(x,t) = e^{t\frac{d}{db}}\left(\frac{1}{\sqrt{2b}}e^{-\frac{x^2}{4b}}\right) = \frac{1}{\sqrt{2(t+b)}}e^{-\frac{x^2}{4(t+b)}}. \quad (35)$$

5. As another example, let us consider the following wave equation

$$(D_t^q)^2 y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t), \quad (-\infty < x < \infty, t > 0) \quad (36)$$

with initial conditions

$$y(x,0) = f(x), \quad D_t^q y(x,0) = g(x). \quad (37)$$

We assume that both $f(x)$ and $g(x)$ vanish as $x \to \pm\infty$. In this case the Fourier-Mellin transformed function $Y^*(\xi,s)$ obeys

$$[s-1]_q[s-2]_q Y^*(\xi,s-2) = -\xi^2 Y^*(\xi,s). \quad (38)$$

The general solution is

$$Y^*(\xi,s) = \left[A(\xi)(-i\xi)^{-s} + B(\xi)(i\xi)^{-s}\right]\Gamma_q(s), \quad (39)$$

where $A(\xi)$ and $B(\xi)$ are some functions of ξ. Performing the inverse-Mellin transform, we get

$$Y(\xi,t) = \frac{1-q}{\ln q^{-1}}\{A(\xi)E_q(iq(1-q)\xi t) + B(\xi)E_q(-iq(1-q)\xi t)\}. \quad (40)$$

Here $Y(\xi,t)$ is the Fourier transform of $y(x,t)$ with respect to x. Now we rewrite (40) in terms of the *q-Sine* and the *q-Cosine* function which are defined by [4]

$$\mathrm{Sin}_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}, \quad (41)$$

$$\mathrm{Cos}_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}. \quad (42)$$

The result is

$$y(\xi,t) = \frac{1-q}{\ln q^{-1}}\{C(\xi)\,\mathrm{Cos}_q(q(1-q)\xi t) + D(\xi)\,\mathrm{Sin}_q(q(1-q)\xi t)\}, \quad (43)$$

where the functions $C(\xi)$ and $D(\xi)$ are linear combinations of $A(\xi)$ and $B(\xi)$. The inverse-Fourier transform of (43) is

$$y(x,t) = \frac{1-q}{\sqrt{2\pi}\ln q^{-1}}\int_{-\infty}^{\infty}\{C(\xi)\,\mathrm{Cos}_q(q(1-q)\xi t) + D(\xi)\,\mathrm{Sin}_q(q(1-q)\xi t)\}e^{-i\xi x}\,d\xi. \quad (44)$$

Letting $t = 0$ in (44), one can check that the function $C(\xi)$ is related to the Fourier transform of $f(x)$ by

$$F(\xi) = \frac{1-q}{\ln q^{-1}}C(\xi). \quad (45)$$

Making use of the following relations, which can be obtained by means of (33):

$$D_t^q\,\mathrm{Sin}_q(\lambda t) = \frac{\lambda}{q(1-q)}\mathrm{Cos}_q(\lambda t), \quad (46)$$

$$D_t^q\,\mathrm{Cos}_q(\lambda t) = -\frac{\lambda}{q(1-q)}\mathrm{Sin}_q(\lambda t), \quad (47)$$

we can relate $D(\xi)$ to the Fourier transform $G(\xi)$ of $g(x)$ as follows:

$$G(\xi) = \frac{1-q}{\ln q^{-1}}D(\xi)\xi. \quad (48)$$

With these results, we finally obtain the solution to the initial problem of Eq. (36):

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{F(\xi)\,\mathrm{Cos}_q(q(1-q)\xi t) + \frac{G(\xi)}{\xi}\mathrm{Sin}_q(q(1-q)\xi t)\right\}e^{-i\xi x}\,d\xi. \quad (49)$$

This solution is the q-analogue of the solution to the corresponding classical case given in [19].

6. We now see how the above steps are generalised to the equation:

$$(D_t^q)^n y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t),$$
$$(-\infty < x < \infty, t > 0,\ n \geq 2) \tag{50}$$

with initial conditions

$$y(x,0) = f(x), \quad (D_t^q)^k y(x,0) = g_k(x),$$
$$k = 1,\ldots,n-1, \tag{51}$$

where the functions $f(x)$ and $g_k(x)$ are assumed to vanish as $x \to \pm\infty$. The Fourier-Mellin transformed function $Y^*(\xi,s)$ obeys

$$(-1)^n [s-1]_q [s-2]_q \cdots [s-n]_q Y^*(\xi,s-n)$$
$$= -\xi^2 Y^*(\xi,s). \tag{52}$$

The general solution is

$$Y^*(\xi,s)$$
$$= \Gamma_q(s) \xi^{-\frac{2s}{n}} \sum_{m=0}^{n-1} A_m(\xi) \left[-e^{-\frac{(2m+1)}{n}\pi i} \right]^s. \tag{53}$$

where $A_m(\xi)$ are some functions of ξ. We can now perform the inverse Mellin and Fourier transforms to get the final solution, which is given formally as

$$y(x,t) = \frac{1-q}{\sqrt{2\pi} \ln q^{-1}} \sum_{m=0}^{n-1} \int_{-\infty}^{\infty} A_m(\xi)$$
$$\times E_q\left(q(1-q) e^{\frac{(2m+1)}{n}\pi i} \xi^{\frac{2}{n}} t \right) e^{-i\xi x}\, d\xi. \tag{54}$$

The functions $A_m(\xi)$ can then be related to the Fourier transforms of the functions $f(x)$ and $g_k(x)$ from the initial conditions.

7. To summarise, we show that the Mellin transform of the q-derivative resembles most closely the corresponding expression in classical analysis, whereas transforms such as the Fourier and the Laplace transform fail in this respect. As such the Mellin transform can be useful in solving certain q-difference equations. We illustrated this fact with a few examples. However, for the Mellin transform to be really useful, a more complete knowledge of the properties of the q-special functions under various integral transforms (Fourier, Laplace, Mellin, etc.) and their inverses has yet to be attained. What is more desirable is to invent integral transforms or q-integral transforms that possess the virtue of the Fourier and the Laplace transform in the classical analysis mentioned in the introduction.

Acknowledgements

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出席國際學術會議心得報告

何俊麟
淡江大學 物理系

1. 會議名稱：XIII International Congress on Mathematical Physics
2. 日期及地點：89年7月17日至7月22日於英國倫敦帝國學院
(Imperial College, London, UK)

3. 心得：

本會是每三年一度有關數學物理的國际大會。從事數学物理方面研究之学者，大都會在此會上发表最新研究成果。更重要的是，許多當代重要的數学物理大師都會在此綜評各個領域的重要進展及未来的方向。

此次大會與會人數估計近千人。代表台灣出席者，除本人外，尚有淡江大学之高賢忠教授，以及交通大学之李仁吉教授。本人在會中发表一篇有関量子力学中准精确可解模型之工作（題目為：Planar Dirac Electron in Coulomb and Magnetic Fields）。

會議期間，本人出席聆聽了多位大師的演講，進而对當前的重要成果及方向有更深入的了解。在會中也認識多位研究領域相近的外國学者，並交換研究心得，獲益匪淺。

此次會議也讓本人有所感觸。數学物理乃物理学門的一個重要分支。然在此次數学物理的重大國际會議中，在近千位出席者竟只有三位台灣的代表。這顯示數学物理在國内仍未能得到应有的重視。希望將来這種情况能有所改善。

何俊麟

Planar Dirac Electron in Coulomb and Magnetic Fields: an example of quasi-exactly solvable models

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1 Introduction

The Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields is discussed [1]. This is connected to the problem of the two-dimensional hydrogen-like atom in the presence of external magnetic field. For weak magnetic fields, the approximate energy values are obtained by semiclassical method. In the case with strong magnetic fields, we present the exact recursion relations that determine the coefficients of the series expansion of wave functions, the possible energies and the magnetic fields. It is found that analytic solutions are possible for a denumerably infinite set of magnetic field strengths. This system thus furnishes an example of the so-called quasi-exactly solvable models [2]. Solutions in the nonrelativistic limit with both attractive and repulsive Coulomb fields are briefly discussed by means of the method of factorization.

2 Motion of Dirac electron in the Coulomb and magnetic fields

The planar Dirac equation for an electron, with mass m and charge $-e$ $(e > 0)$, minimally coupled to an external electromagnetic field A_μ has the form (we set $c = \hbar = 1$)

$$(i\partial_t - H_D)\Psi(t, \mathbf{r}) = 0, \tag{1}$$

where

$$H_D = \alpha\mathbf{P} + \beta m - eA^0 \equiv \sigma_1 P_2 - \sigma_2 P_1 + \sigma_3 m - eA^0 \tag{2}$$

and $P_k = -i\partial_k + eA_k$.

We shall solve for both positive and negative energy solutions of the Dirac equation (1) and (2) in an external Coulomb field and a constant homogeneous magnetic field $B > 0$ along the z direction:

$$A^0(r) = Ze/r \ (e > 0), \quad A_x = -By/2, \quad A_y = Bx/2\ . \tag{3}$$

We assume the wave functions to have the form

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(-iEt)\psi_l(r, \varphi)\ , \tag{4}$$

where E is the energy of the electron, and

$$\psi_l(r, \varphi) = \frac{1}{\sqrt{r}} \begin{pmatrix} F(r)e^{il\varphi} \\ G(r)e^{i(l+1)\varphi} \end{pmatrix} \tag{5}$$

with integral number l. The function $\psi_l(r, \varphi)$ is an eigenfunction of the conserved total angular momentum $J_z = L_z + S_z = -i\partial/\partial\varphi + \sigma_3/2$ with eigenvalue $j = l + 1/2$. The Dirac equation becomes:

$$\frac{dF}{dr} - \left(\frac{l+\frac{1}{2}}{r} + \frac{eBr}{2}\right)F + \left(E + m + \frac{Z\alpha}{r}\right)G = 0\ , \tag{6}$$

$$\frac{dG}{dr} + \left(\frac{l+\frac{1}{2}}{r} + \frac{eBr}{2}\right)G - \left(E - m + \frac{Z\alpha}{r}\right)F = 0\ , \tag{7}$$

where $\alpha \equiv e^2 = 1/137$ is the fine structure constant.

In a strong magnetic field the asymptotic solutions of $F(r)$ and $G(r)$ have the forms $\exp(-ar^2/2)$ with $a = eB/2$ at large r, and r^γ with

$$\gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2} \tag{8}$$

for small r. One must have $Z\alpha < 1/2$, otherwise the wave function will oscillate as $r \longrightarrow 0$ when $l = 0$ and $l = -1$. In this paper we shall look for solutions of $F(r)$ and $G(r)$ which can be expressed as a product of the asymptotic solutions (for small and large r) and a series in the form

$$F(r) = r^\gamma \exp(-ar^2/2) \sum_{n=0} \alpha_n r^n \;, \tag{9}$$

$$G(r) = r^\gamma \exp(-ar^2/2) \sum_{n=0} \beta_n r^n \;, \tag{10}$$

with $\alpha_0 \neq 0$, $\beta_0 \neq 0$. Substituting (9) and (10) into (6) and (7), we obtain

$$\left[\gamma - \left(l + \frac{1}{2}\right)\right] \alpha_0 \; + \; Z\alpha\beta_0 = 0 \;, \tag{11}$$

$$\left[(\gamma+1) - \left(l + \frac{1}{2}\right)\right] \alpha_1 \; + \; Z\alpha\beta_1 + (E+m)\,\beta_0 = 0 \;, \tag{12}$$

$$\left[(n+\gamma) - \left(l + \frac{1}{2}\right)\right] \alpha_n \; + \; Z\alpha\beta_n + (E+m)\,\beta_{n-1} - 2a\alpha_{n-2} = 0 \quad (n \geq 2) \tag{13}$$

from (6), and

$$\left(\gamma + l + \frac{1}{2}\right)\beta_0 \; - \; Z\alpha\alpha_0 = 0 \;, \tag{14}$$

$$\left(n + \gamma + l + \frac{1}{2}\right)\beta_n \; - \; Z\alpha\alpha_n - (E-m)\,\alpha_{n-1} = 0 \quad (n \geq 1) \tag{15}$$

from (7).

Eq.(11) and (14) allow us to express β_0 in terms of α_0 in two forms:

$$\beta_0 = \frac{Z\alpha}{\gamma + l + \frac{1}{2}}\,\alpha_0 \tag{16}$$

$$= -\frac{\gamma - l - \frac{1}{2}}{Z\alpha}\,\alpha_0\,, \tag{17}$$

which are equivalent since $\gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2}$. Solving (12) and (15) with $n = 1$ gives

$$\alpha_1 = -\frac{\left(\gamma + l + \frac{1}{2}\right)(E-m) + \left(\gamma + l + \frac{3}{2}\right)(E+m)}{(2\gamma+1)\left(\gamma + l + \frac{1}{2}\right)}\,Z\alpha\;\alpha_0, \tag{18}$$

$$\beta_1 = \frac{2(\gamma + l)\,E - m}{(2\gamma+1)}\,\alpha_0\,. \tag{19}$$

From (15) one sees that β_n $(n \geq 1)$ are obtainable from α_n and α_{n-1}. To determine the recursion relations for the α_n, we simply eliminate β_n and β_{n-1} in (13) by means of (15). This leads to (for $n \geq 2$):

$$\begin{aligned}
&\left(n+\gamma+l-\frac{1}{2}\right)\left(n^2+2n\gamma\right)\alpha_n \\
&+Z\alpha\left[\left(n+\gamma+l-\frac{1}{2}\right)(E-m)+\left(n+\gamma+l+\frac{1}{2}\right)(E+m)\right]\alpha_{n-1} \\
&+\left(n+\gamma+l+\frac{1}{2}\right)\left[E^2-m^2-2a\left(n+\gamma+l-\frac{1}{2}\right)\right]\alpha_{n-2} = 0
\end{aligned} \tag{20}$$

We impose the sufficient condition that the series parts of $F(r)$ and $G(r)$ should terminate appropriately in order to guarantee normalizability of the eigenfunctions. It follows from (20) that the solution of $F(r)$ becomes a polynomial of degree $(n-1)$ if the series given by (20) terminates at a certain n when $\alpha_n = \alpha_{n+1} = 0$, and $\alpha_m = 0$ $(m \geq n+2)$ follow from (20). Then from (15) we have $\beta_{n+1} = \beta_{n+2} = \ldots = 0$. Thus in general the polynomial part of the function $G(r)$ is of one degree higher than that of F. Now suppose we have calculated α_n in terms of α_0 ($\alpha_0 \neq 0$) from (18) and (20) in the form:

$$\alpha_n = K(l, n, E, a, Z)\ \alpha_0\ . \tag{21}$$

Then two conditions that ensure $\alpha_n = 0$ and $\alpha_{n+1} = 0$ are

$$K(l, n, E, a, Z) = 0 \tag{22}$$

and

$$E^2 - m^2 = 2a\left(n + \gamma + l + \frac{1}{2}\right)\quad ,\quad n = 1, 2, \ldots \tag{23}$$

Since the right hand side of (23) is always non-negative [1], we must have $|E| \geq m$ for the energy.

For any integer n, eqs.(22) and (23) give us a certain number of pairs (E, a) of energy E and the corresponding magnetic field B (or a) which would guarantee normalizability of the wave function. Thus only parts of the whole spectrum of the system are exactly solved. The system can therefore be considered as an example of the quasi-exactly solvable models defined in [2]. In principle the possible values of E and a can be obtained by first expressing the a (or E) in (22) in terms of E (a) according to (23). This gives an algebraic equation in E (a) which can be solved for real E (a). The corresponding values of a (E) are then obtained from (23). In practice the task could be tedious. The simplest cases, namely, those with $n = 1,\ 2$ and 3, were discussed in detail in [1].

[1] For $l \geq 0$, this is obvious. For $l \leq -1$, one has $-1/2 \leq \gamma + l + \frac{1}{2} \leq 0$, recalling that $Z\alpha < 1/2$.

3 Non-relativistic limit and method of factorization

The electron in 2+1 dimensions in the nonrelativistic approximation is described by one-component wave function ψ satisfying

$$i\frac{\partial\psi}{\partial t} = \left(\frac{P_1^2 + P_2^2}{2m} + \frac{eB}{2m} - \frac{Ze^2}{r}\right)\psi \ , \tag{24}$$

where, as before, $P_k = -i\partial_\mu + eA_\mu$.

One can proceed in the same manner as in the Dirac case to solve for the possible energies and magnetic fields. However, in this case one can also follow a method closely resembling the method of factorization in nonrelativistic quantum mechanics. We shall discuss this method briefly below. Both the attractive and repulsive Coulomb interactions will be considered, since planar two electron systems in strong external homogeneous magnetic field (perpendicular to the plane in which the electrons is located) are also of considerable interest for the understanding of the fractional quantum Hall effect. Let us assume

$$\psi(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(-iEt + il\varphi) r^{|l|} \exp(-ar^2/2) Q(r) \ , \tag{25}$$

where Q is a polynomial, and $a = eB/2$ as defined before. Substituting (25) into (24), we have

$$\left[\frac{d^2}{dx^2} + \left(\frac{2\gamma}{x} - x\right)\frac{d}{dx} + \left(\epsilon \pm \frac{b}{x}\right)\right] Q(x) = 0 \ , \tag{26}$$

Here $x = r/l_B$, $l_B = 1/\sqrt{eB}$, $\gamma = |l| + 1/2$, $b = 2m|Z|\alpha l_B = |Z|\alpha\sqrt{2m/\omega_L}$, and $\epsilon = E/\omega_L - (2 + l + |l|)$. The upper (lower) sign in (26) corresponds to the case of attractive (repulsive) Coulomb interaction.

It is seen that the problem of finding spectrum for (26) is equivalent to determining the eigenvalues of the operator

$$H = -\frac{d^2}{dx^2} - \left(\frac{2\gamma}{x} - x\right)\frac{d}{dx} \mp \frac{b}{x} \,. \tag{27}$$

We want to factorize the operator (27) in the form

$$H = a^+ a + p, \tag{28}$$

where the quantum numbers p are related to the eigenvalues of (26) by $p = \epsilon$. The eigenfunctions of the operator H at $p = 0$ must satisfy the equation

$$a\psi = 0 \,. \tag{29}$$

Suppose polynomial solutions exist for (26), say $Q = \prod_{k=1}^{s}(x - x_k)$, where x_k are the zeros of Q, and s is the degree of Q. Then the operator a must have the form

$$a = \frac{\partial}{\partial x} - \sum_{k=1}^{s} \frac{1}{x - x_k} \,, \tag{30}$$

and the operator a^+ has the form

$$a^+ = -\frac{\partial}{\partial x} - \frac{2\gamma}{x} + x - \sum_{k=1}^{s} \frac{1}{x - x_k} \,. \tag{31}$$

Substituting (30) and (31) into (28) and then comparing the result with (27), we obtain the following set of equations for the zeros x_k (the so-called Bethe *ansatz* equations [2]):

$$\frac{2\gamma}{x_k} - x_k - 2\sum_{j\neq k}^{s} \frac{1}{x_j - x_k} = 0 \,, \quad k = 1, \ldots, s \,, \tag{32}$$

as well as the two relations:

$$\pm b = 2\gamma \sum_{k=1}^{s} x_k^{-1} \,, \quad s = p \,. \tag{33}$$

Summing all the s equations in (32) enables us to rewrite the first relation in (33) as

$$\pm b = \sum_{k=1}^{s} x_k \ . \tag{34}$$

From these formulas we can find the simplest solutions as well as the values of energy and magnetic field strength. The second relation in (33) gives $E = \omega_L(2 + s + l + |l|)$.

For $s = 1, 2$ the zeros x_k and the values of the parameter b for which solutions in terms of polynomial of the corresponding degrees exist can easily be found from (32) and (34) in the form

$$\begin{aligned} s = 1\,, \quad x_1 &= \pm\sqrt{2|l| + 1}\,, \quad b = \sqrt{2|l| + 1}\,, \\ s = 2\,, \quad x_1 &= (2|l| + 1)/x_2\,, \quad x_2 = \pm(1 + \sqrt{4|l| + 3})/\sqrt{2}\,, \\ b &= \sqrt{2(4|l| + 3)}\ . \end{aligned} \tag{35}$$

From (35) and the definition of b one has the corresponding values of magnetic field strengths

$$\begin{aligned} \omega_L &= 2m\frac{(Z\alpha)^2}{2|l| + 1}, \quad s = 1\,, \\ \omega_L &= m\frac{(Z\alpha)^2}{4|l| + 3}, \quad s = 2\,, \end{aligned} \tag{36}$$

as well as the energies

$$\begin{aligned} E_1 &= \frac{2m(Z\alpha)^2}{2(2|l| + 1)}(3 + l + |l|)\,, \\ E_2 &= \frac{m(Z\alpha)^2}{(4|l| + 3)}(4 + l + |l|)\ . \end{aligned} \tag{37}$$

The corresponding polynomials are

$$\begin{aligned} Q_1 &= x - x_1 = x \mp b\,, \\ Q_2 &= \prod_{k=1}^{2}(x - x_k) = x^2 \mp bx + 2|l| + 1\ . \end{aligned} \tag{38}$$

The wave functions are described by (25). For $s = 1, 2$ for the repulsive Coulomb field the wave functions do not have nodes (for $|l| = 0, 1$), i.e. the states described by them are ground states, while for the attractive Coulomb field the wave function for $s = 1$ has one node (first excited state) and the wave function for $s = 2$ has two nodes (second excited state).

References

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