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摘　　要

在本計劃補助下，我们完成一項有関相对性電子在二維庫倫場下的精確解問題的研究。我们也討論了此系統在庫倫場添達臨界值时之不穩定性。

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ABSTRACT

Under the support of this NSC grant, we completed an investigation on the exact solutions of relativistic electron in a two-dimensional Coulomb electric field. We had also considered the instability problem of the system when the charge of the Coulomb field exceeds certain critical value.

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DIRAC ELECTRON IN A COULOMB FIELD IN $(2+1)$ DIMENSIONS

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Exact solutions of Dirac equation in two spatial dimensions in the Coulomb field are obtained. Equation which determines the so-called critical charge of the Coulomb field is derived and solved for a simple model.

1. Introduction

Planar nonrelativistic electron systems in a uniform magnetic field are fundamental quantum systems which have provided insights into many novel phenomena, such as the quantum Hall effect, the theory of anyons and particles obeying fractional statistics.[1,2] On the other hand, planar electron systems with energy spectrum described by the Dirac Hamiltonian have also been studied as field-theoretic models for the quantum Hall effect and anyon theory.[3,4] Related to these field-theoretic models are the recent interesting studies regarding the instability of the naive vacuum and spontaneous magnetization in $(2+1)$-dimensional quantum electrodynamics (QED), which is induced by a bare Chern–Simons term.[5] In view of these developments, it is essential to have a better understanding of the properties of planar Dirac particles in the presence of external electromagnetic fields.

In this letter we would like to consider solutions of Dirac equation in two spatial dimensions in the presence of a strong Coulomb field, and to discuss instability of the Dirac vacuum in a regulated strong Coulomb field. In three space dimensions the effect of positron production by strong Coulomb field was predicted in Ref. 6 and studied in Refs. 7–15.

2. Motion of an Electron in the Coulomb Field

Let us consider a relativistic electron in two spatial dimensions in a Coulomb field, the vector potential of which is specified as

$$A^0(r) = -Ze/r\,, \qquad A^x = A^y = 0\,. \tag{1}$$

In $(2+1)$ dimensions, the Dirac matrices may be represented in terms of the Pauli matrices. We choose $\boldsymbol{\alpha} = (-\sigma^2, \sigma^1)$ and $\beta = \sigma^3$, then the Dirac equation has the form $(c = \hbar = 1)$

$$(i\partial_t - H_D)\Psi = 0\,, \tag{2}$$

where

$$H_D = \boldsymbol{\alpha}\cdot\mathbf{P} + \beta m + eA^0 \equiv \sigma^1 P_2 - \sigma^2 P_1 + \sigma^3 m + eA^0\,, \tag{3}$$

is the Dirac Hamiltonian, $P_\mu = i\partial_\mu - eA_\mu$ is the operator of generalized momentum of electron, m is the rest mass of the electron, and $e = -e_0$, $e_0 > 0$ is its electric charge. The conserved total angular momentum only has a single component, namely, $J_z = L_z + S_z$, where $L_z = -i\partial/\partial\varphi$ and $S_z = \sigma^3/2$.

We shall look for solutions of (2) in the form

$$\Psi(t,\mathbf{x}) = \frac{1}{\sqrt{2\pi}}\exp(-i\varepsilon Et)\psi(r,\varphi)\,, \tag{4}$$

where $\varepsilon = \pm 1$ and $E > 0$. We assume the ansatz

$$\psi(r,\varphi) = e^{il\varphi}\begin{pmatrix} f(r) \\ g(r)e^{i\varphi}\end{pmatrix}\,, \tag{5}$$

where l is an integer. The function $\psi(r,\varphi)$ is an eigenfunction of the total angular momentum J_z with eigenvalue $l+1/2$. Substituting (4) and (5) into (2), and taking into account the equations

$$P_x \pm iP_y = -ie^{\pm i\varphi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial\varphi}\right)\,, \tag{6}$$

we obtain

$$\begin{aligned} &\frac{df}{dr} - \frac{1}{r}f + \left(\varepsilon E + m + \frac{Z\alpha}{r}\right)g = 0\,, \\ &\frac{dg}{dr} + \frac{1+l}{r}g - \left(\varepsilon E - m + \frac{Z\alpha}{r}\right)f = 0\,, \end{aligned} \tag{7}$$

where $\alpha \equiv e^2 = 1/137$ is the fine structure constant.

The exact solutions and the energy eigenvalues with $\varepsilon E < m$ corresponding to the stationary states of the Dirac equation may be found in full analogy with the case of three space dimensions (we shall follow Ref. 16). Let us look for functions f and g in the form

$$\begin{aligned} f &= \sqrt{m+E}\,e^{-\rho/2}\rho^{\gamma-1}(Q_1 + Q_2)\,, \\ g &= \sqrt{m-E}\,e^{-\rho/2}\rho^{\gamma-1}(Q_1 - Q_2)\,, \end{aligned} \tag{8}$$

where

$$\rho = 2\lambda r\,, \qquad \lambda = \sqrt{m^2 - E^2}\,, \qquad \gamma = \frac{1}{2} + \sqrt{(l+1/2)^2 - (Z\alpha)^2}\,. \tag{9}$$

The value of γ is to be found by studying the behavior of the wave function at small r. From (7) and (8), together with the equality $(\gamma - 1/2)^2 - (Z\alpha E/\lambda)^2 = (l+1/2)^2 - (Z\alpha m/\lambda)^2$, one can derive the differential equations satisfied by Q_1 and Q_2. It turns out that the functions Q_1 and Q_2 which rendered the solutions of (7) finite at $\rho = 0$ are given in terms of the confluent hypergeometric function $F(a,b;z)$ as

$$\begin{aligned} Q_1 &= AF\left(\gamma - \frac{1}{2} - \left(\frac{Z\alpha E}{\lambda}\right), 2\gamma; \rho\right)\,, \\ Q_2 &= BF\left(\gamma + \frac{1}{2} - \left(\frac{Z\alpha E}{\lambda}\right), 2\gamma; \rho\right)\,. \end{aligned} \tag{10}$$

The constants A and B are related by

$$B = \frac{\gamma - \frac{1}{2} - \frac{Z\alpha E}{\lambda}}{l + \frac{1}{2} + \frac{Z\alpha m}{\lambda}}A\,. \tag{11}$$

The energy eigenvalues are defined by

$$\gamma - \frac{1}{2} - \frac{Z\alpha E}{\lambda} = -n_r\,. \tag{12}$$

It is easy to show that the following values of the quantum number n_r are allowed: $n_r = 0, 1, 2, \ldots$, if $l \geq 0$, and $n_r = 1, 2, 3, \ldots$ if $l < 0$. Therefore, the electron energy spectrum in the Coulomb field (1) has the form

$$E = m\left[1 + \frac{(Z\alpha)^2}{\left(n_r + \sqrt{(l+1/2)^2 - (Z\alpha)^2}\right)^2}\right]^{-1/2}\,. \tag{13}$$

It is seen that

$$E_0 = m\sqrt{1 - (2Z\alpha)^2} \tag{14}$$

for $l = n_r = 0$, and E_0 becomes zero at $Z\alpha = 1/2$, whereas in three spatial dimensions, E_0 equals zero at $Z\alpha = 1$. Thus, in two space dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ has no physical meaning at a much lower value of $Z\alpha = 1/2$, and the corresponding solution of the Dirac equation oscillates near the point $r \to 0$.

3. Critical Charge

It is known[14,16] that in three spatial dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ becomes purely imaginary when $Z > 137$, and that its interpretation as electron energy has no physical meaning. To determine the electron energy spectrum in the Coulomb field with such a charge we need to eliminate the singularity of the Coulomb potential of a point-charge at $r = 0$ by cutting off the Coulomb potential at small distances. This is equivalent to taking into account the nucleus size. In three space dimensions the electron energy spectrum in the Coulomb field regulated at small distances was first considered in Ref. 15. With increasing Z in the region $Z > 137$, the electron energy levels in such a field were found to decrease, become negative, and may cross the boundary of the lower energy continuum, $E = -m$. The value of $Z|e| = Z_c|e|$ at which the lowest electron energy level reach the boundary of the lower energy continuum is called the critical charge for the electron ground state.[12-14] If Z continues to grow and enters the transcritical region with $Z > Z_c$, the lowest electron energy level "sinks" into the lower energy continuum, which result in a rearrangement of the vacuum of the QED. This rearrangement is constrained by Pauli's exclusion principle. If the electron ground state at $Z < Z_c$ is vacant, two electron–positron pairs are created; if it is half-occupied, one pair is created; and if it is occupied, no pairs are created. The Coulomb potential is repulsive for the created positrons, so they go to infinity. Hence at $Z > Z_c$ a quasistationary state appears in the lower energy continuum and the new vacuum of QED, which corresponds to filling of all the electron states with $E < -m$, has the total electric charge $2e$.[12-14] Indeed, all the electron states with $E < -m$ (the Dirac sea) were filled at $Z < Z_c$, so electrons created by the strong Coulomb field with $Z > Z_c$ cannot be described by means of a convenient wave function, and the notion of charged vacuum was introduced to describe these states.[7-9,12,13] In terms of the new vacuum, the density of electric charge $\rho(r)$ is classical. It is a function characterizing the spatial distribution of the real electric charge appearing in the new (charged) vacuum, while in terms of the old (uncharged) vacuum, this function should be interpreted as the probability of two electrons (with charge $2e$) being present at a given point in space.

We would like to see how the same system behaves in two dimensions. Let us therefore consider the solutions and the energy eigenvalues corresponding to stationary states of the Dirac equation in the Coulomb field with $2Z > 137$ and find the corresponding value of Z_c. To find Z_c it is enough to study the energy region near the boundary of the lower energy continuum, $-m$. We shall rewrite the Dirac equation, taking into account the fact that $\varepsilon E \approx -m$. Introducing functions $F(r) = rf(r)$ and $G(r) = rg(r)$, and eliminating $G(r)$ from (7), we arrive at the equation for the function F near the boundary of the lower energy continuum $-m$ in the form

$$\frac{d^2F(r)}{dr^2} + \left(E^2 - m^2 + \frac{2\varepsilon EZ\alpha}{r} + \frac{(Z\alpha)^2 - l(l+1)}{r^2}\right)F(r) = 0\,. \tag{15}$$

We note that near the boundary of the upper energy continuum for $\varepsilon E \approx m$, the function $G(r)$ obeys Eq. (15) with $F(r)$ replaced by $G(r)$.

Solution of (15), which tends to zero as $r \to \infty$, may be expressed by means of the Whittaker function (see also Ref. 17)

$$F(r) \sim W_{\beta, i\frac{\nu}{2}}(2\lambda r)\,, \tag{16}$$

where

$$\beta = \frac{\varepsilon EZ\alpha}{\lambda}\,, \qquad \nu = 2\sqrt{(Z\alpha)^2 - (l+1/2)^2}\,, \qquad \lambda = \sqrt{m^2 - E^2}\,. \tag{17}$$

From (7), the function $G(r)$ at $\varepsilon E = -m$ can be obtained as

$$G(r) = \frac{1}{Z\alpha}\left((1+l)F - r\frac{dF}{dr}\right)\,. \tag{18}$$

Near the boundary of the upper energy continuum, the function $G(r)$ is given by the Whittaker function in Eq. (16) with $\varepsilon E = m$, while the function $F(r)$ can be found from the relation

$$F(r) = \frac{1}{Z\alpha}\left(lG + r\frac{dG}{dr}\right)\,. \tag{19}$$

Using the asymptotic representation for Whittaker function at large $|z|$ in the form

$$W_{\beta,\mu}(z) \sim e^{-z/2}(z)^\beta\,, \tag{20}$$

it is seen that the bound electron state (with $|E| < m$ or $\lambda > 0$) is localized in the plane.

If we treat (15) as a one-dimensional Schrödinger-type equation which describes a particle with "particle energy" $E' = (E^2 - m^2)/2m$ in the field of the effective potential (in particular, for $l = 0$), the behavior mentioned above may be easily understood:

$$U_{\rm eff}(r) = -\varepsilon EZ\alpha/mr - (Z\alpha)^2/2mr^2\,.$$

We note that the effective potential is wide enough near the boundary of the lower energy continuum (for behavior of the effective potential in three space dimensions see e.g. Ref. 14), and that the effective potential in two space dimensions does not contain the spin electron term $-s(s+1) \equiv -3/4$.

The solution of (15) at $\varepsilon E = -m$ can be written in terms of the MacDonald function of imaginary order

$$F(r) = \sqrt{r}K_{i\nu}(\sqrt{8mZ\alpha r})\,. \tag{21}$$

The function $G(r)$ at $\varepsilon E = -m$ is determined by (18). In the following we shall determine the critical value Z_c for a simple model in which the potential $A_0(r)$ is regulated at small distances as follows:

$$A_0^Z(r) = \begin{cases} -\dfrac{Ze}{r}\,, & r \geq R\,, \\[2ex] -\dfrac{Ze}{R}\,, & r \leq R\,. \end{cases} \tag{22}$$

In the region $r \leq R$, the function $F(r)$ obeys the equation

$$\frac{d^2R}{dr^2} - \frac{dF}{rdr} + \left(\left(\varepsilon E + \frac{Z\alpha}{R}\right)^2 - m^2 + \frac{1-l^2}{r^2}\right) F(r) = 0\,. \tag{23}$$

The solution of (23) is

$$F(r) = r(A_1 J_{|l|}(\kappa r) + B_1 Y_{|l|}(\kappa r))\,, \tag{24}$$

where

$$\kappa = \sqrt{\left(\varepsilon E + \frac{Z\alpha}{R}\right)^2 - m^2}\,, \tag{25}$$

and $J_n(z)$ and $Y_n(z)$ are the Bessel and the Neumann functions of integer order n.

In order for the function $F(r)$ to be finite at the point $r = 0$, we need to set $B_1 = 0$. To determine the energy spectrum we need to match the solutions at the point $r = R$:

$$\left(\frac{G(r)}{F(r)}\right)_{r=R-0} = \left(\frac{G(r)}{F(r)}\right)_{r=R+0}\,. \tag{26}$$

Taking into account the fact that R is much less than $1/m$ so that $\kappa \approx Z\alpha/R$, we obtain, for the state with $l = 0$ and $\varepsilon E = -m$, the following equation that determines (at fixed R) the critical charge:

$$\frac{J_1(X)}{J_0(X)} = \frac{1}{2X}\left(1 - \sqrt{z}\frac{K'_{i\nu}(z)}{K_{i\nu}(z)}\right)\,. \tag{27}$$

Here $X = Z_c\alpha$, $\nu = \sqrt{4X^2-1}$, $z = \sqrt{8mRX}$ and $K'_{i\nu}(z) = dK_{i\nu}(z)/dz$. Equation (27) may be solved numerically. As we are interested only in the critical charge corresponding to the ground state, we can consider small values of z. In this case, the MacDonald function with imaginary order $K_{i\nu}(z)$ has the following expansion:

$$K_{i\nu}(z) \longrightarrow \sqrt{\frac{\pi}{\nu \sinh \pi\nu}}\left[\sin\left(\nu \ln\frac{2}{z} + \arg\Gamma(1+i\nu)\right) + \frac{z^2}{4\sqrt{1+\nu^2}}\sin\left(\nu\ln\frac{2}{z} + \arg\Gamma(1+i\nu) + \tan^{-1}\nu\right) + \cdots\right]. \tag{28}$$

Numerical solutions of Eq. (27) give $Z_c \approx 84$, 89 at $Rm = 0.02$ and 0.03, respectively. For comparison purpose, we recall that $Z_c \approx 170$ at $Rm = 0.03$ for the analogical model in three space dimensions.[12-14]

Thus, the Dirac vacuum in two space dimensions in the presence of a strong Coulomb field is unstable against electron–positron production at significantly smaller values of the critical charge than in the case of three spatial dimensions.

Another difference between these two cases results from the fact that electrons confined to a plane behave like a spinless fermion. So if the ground electron state at $Z < Z_c$ is vacant, one pair is created; if it is occupied, no pairs are created.

4. Summary

In this letter we present the exact solutions of the $(2+1)$-dimensional Dirac equation with a Coulomb field, and determine the critical charge Z_c of a regulated Coulomb source for which the Dirac vacuum of the system become unstable. At $Z > Z_c$ the lowest electron state of discrete spectrum is the state with $n_r \neq 0$. So if the electron ground state at $Z < Z_c$ was vacant, then at $Z > Z_c$ an electron would be created, together with a hole in the lower energy continuum. According to Dirac, this hole is to behave as a real positive charged particle far from the Coulomb center. Thus, phenomena that may occur at $Z > Z_c$ are many-particle, and to describe them it is necessary to apply the quantum field theory. From the point of view of QED, the strong Coulomb field with $Z > Z_c$ creates a positron and changes the vacuum in such a way that it gains the electric charge which is exactly equal to the electron charge e. The spatial distribution of the electric charge appearing in the vacuum looks like the spatial distribution of the electron charge in the level with $n_r = 0$ in an atom with $Z < Z_c$. However, the density of the vacuum electric charge is a function characterizing the spatial distribution of the real electric charge appearing in the vacuum, while in the atom this function gives the probability density that the electron (with charge e) may be found at a given point in space.

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