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毀約風險與保險需求之研究

Default Risk and Insurance Demand

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一、 中文摘要

本文提出「平均全額保險」之概念，其定義為，以不可投保風險之平均值取代不可投保風險本身之隨機變數後，使得可投保風險消除之保險額度。利用可回歸性之假設，本文導出，全額保險、超額保險與不足額保險時，背景風險與可投保風險之相關性及保險的不公平性間關係之條件。然後把這些條件與毀約風險存在時之條件比較。本文導出之結果，可用來解釋文獻中提出的背景風險與毀約風險之差異。

Abstract

In this paper, "full-insurance coverage on average" is defined as the coinsurance rate that eliminates all insurable risk when the uninsurable risk is evaluated at its mean. Using the regressibility assumption, this paper derives the conditions on the correlation between background and insurable risks and the actuarial unfairness of insurance under which full-, over- or under-insurance on average is optimal. These conditions are compared to those for the case of default risk. Together they explain intuitively the different results under the cases of background risk and default risk obtained in the literature.

Keywords: Default Risk, Incomplete Insurance Markets, Additive Idiosyncratic Risk, Full Insurance Coverage on Average

1. 緣由與目的

In his seminal paper, Arrow [1] shows that if a risk averse individual is offered an insurance against a random loss at an actuarially fair premium rate, she will choose to buy one-hundred percent coverage.[1] Recent research, however, indicates that this result does not hold in general when insurance markets become incomplete. Here, market incompleteness means that individuals are facing some risks that are uninsurable due to the absence of the corresponding insurance markets possibly as a result of moral hazard or adverse selection.

By using a simple model with only four states of nature, Doherty and Schlesinger [5] show that full-insurance coverage may not be optimal when an individual's wealth consists of not only an insurable random loss (say medical expenditure) but also an uninsurable idiosyncratic (background) risk (say income risk). They show that, under their simple model, traditional theory holds only

[1] An insurance is actuarially fair (favorable/unfavorable) when the premium payment is equal to (smaller than/higher than) the expected value of the loss. Mossin [10] and Smith [15] show that actuarial favorableness (unfairness) has a positive (negative) effect on insurance coverage.

when the insurable and uninsurable risks are statistically independent. Moreover, the correlation between the two risks seems to have a monotonic relation with insurance coverage.[2]

More recently, Doherty and Schlesinger [7] study the effect of the presence of an uninsurable risk of insurer's default on insurance coverage. They find surprisingly that, even in their simple model with only three states of nature in which the default risk and the insurable risk are independent, optimal coinsurance rate can be equal to, greater than or less than one when insurance is fair. Unfortunately, the authors do not provide an intuitive explanation for the apparent difference between the case of an idiosyncratic risk and the case of the risk of insurer's default. So far, very little progress has been made in the theory of insurance demand for the case of default risk.[3]

Careful inspection of the incomplete insurance market literature reveals that there is a lack of an appropriate definition for "full-insurance coverage." In general, when an uninsurable risk exists, a coinsurance rate of one on the insurable risk does not eliminate all risk from the individual. Therefore, it is not always meaningful to use the coinsurance rate of one as the definition for "full insurance" and to compare it with an individual's optimal coinsurance rate.[4] As a consequence, before comparing the case of uninsurable background risk and the case of default risk, we need to have a meaningful and workable definition for "full-insurance coverage." To this end, this paper introduces the concept of "full-insurance on average" which is defined as the coinsurance rate at which all risk is eliminated when the uninsurable risk is evaluated at its mean.

The main purpose of this paper is to check whether "full-insurance on average" is optimal under the case of idiosyncratic risk and the case of default risk. A novelty of this paper is that no specific distributions for the insurable and uninsurable risks are assumed. The only restriction being imposed is the "regressibility assumption" employed in the literature of indirect hedging of exchange rate risk (see, e.g., Broll, Wahl, and Zilcha [2] and Broll and Wahl [3]). It turns out that this assumption simplifies the analysis substantially. Moreover, it will be shown that this assumption is a generalization of the bivariate normal and the perfect correlation assumptions used by Doherty and Schlesinger (1983b, 1985). It will also be shown in a footnote that the same results can be reached with the regressibility assumption replaced by the "small-risk assumption" together with second-order Taylor's approximation.[5]

This paper is organized as follows. In Section 2, a model of insurance demand under uninsurable background risk is analyzed. It is

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[2] They show that, under their four-state model, when the insurable and the background risks are positively (negatively) correlated, optimal coinsurance rate is greater (less) than one. Schlesinger and Doherty [13] derive similar results under a more general model in which the background risk and the insurable risk are perfectly correlated. Eeckhoudt and Kimball [8] show that traditional theory holds in the case of an independent background risk. Doherty and Schlesinger [6] show that optimal deductible equal zero when the two risks have a bivariate normal distribution.

[3] Only very special cases are considered in the literature. For example, Schlesinger and Schulenburg [14] consider the relation between Arrow-Pratt risk aversion and insurance demand when a total default is possible.

[4] Doherty and Schlesinger (1983a, 1990), Schlesinger and Doherty [13], and Eeckhoudt and Kimball [8] compare an individual's optimal coinsurance rate with the coinsurance rate of one even though they realize that a coinsurance rate of one cannot eliminate all risk faced by the insured.

[5] The second-order Taylor approximation that can be employed is a two-random-variable version of the one-random-variable Taylor approximation used extensively in the literature of uncertainty (see, e.g., Pratt [11] and Leland [9]). As suggested by Samuelson [12], when the risks considered have "small-risk distributions," one is justified to use low-order Taylor approximation to analyze the individual's portfolio selection problem.

found that, under the regressibility assumption, when the background risk and the insurable risk are uncorrelated, full- (over-/under-) insurance *on average* is optimal if, and only if insurance is fair (favorable/unfavorable). A negative (positive) correlation between the two risks, however, implies that under-insurance (over-insurance) on average is optimal whenever insurance is actuarially fair or unfavorable (fair or favorable). The former result agrees with that of Eeckhoudt and Kimball (1992) while the latter result complements theirs. In Section 3, a model of insurer's default is analyzed under the regressibility assumption. Full insurance *on average* in this case requires the coinsurance rate to be larger than one. It is found that, when the insurable and the default risks are independent, under-insurance on average is optimal if insurance is fair or unfavorable. When the two risks are not independent, the relations between optimal coinsurance rate, loading charges, and the correlation coefficient of the insurable and the default risks are derived. It turns out that knowing the sign of the correlation is insufficient for determining whether full- (under-/over-) insurance on average is optimal even when insurance is fair. These results provide an intuitive explanation for the ambiguous results obtained by Doherty and Schlesinger [7]. Section 4 concludes.

2. Additive Idiosyncratic Risk and Insurance Coverage

The model analyzed in this section is a variation of that of Eeckhoudt and Kimball [8]. An individual with initial wealth w is facing an insurable loss $\tilde{z}$ which is a random variable with positive realizations, mean $\bar{z}$, and standard deviation σ_z. She can choose a coinsurance rate α to insure against the potential loss by paying a premium. Let λ be the constant loading factor of the insurance so that the premium payment equals $\alpha(1+\lambda)\bar{z}$. The insurance is actuarially fair (favorable/unfavorable) if, and only if $\lambda = (</>) 0$. In addition to the insurable risk, the individual is also facing an additive uninsurable background risk $\tilde{y}$ which is a random variable with mean $\bar{y}$ and standard deviation σ_y. The background risk reduces (increases) the individual's wealth when its realization is positive (negative).[6] Assume that the joint distribution of $\tilde{z}$ and $\tilde{y}$ is known to the insured while the distribution of $\tilde{z}$ is known to the insurance company. The individual's random wealth level is given by

$$w - \tilde{y} - (1-\alpha)\tilde{z} - \alpha(1+\lambda)\bar{z}. \qquad (1)$$

Her problem is to choose α to maximize her von Neumann-Morgenstern expected utility

$$EU = EU[w - \tilde{y} - (1-\alpha)\tilde{z} - \alpha(1+\lambda)\bar{z}], \qquad (2)$$

where U is assumed to follow standard assumptions with $U' > 0$ and $U'' < 0$. The first-order condition for an optimum is given by

$$(dEU/d\alpha)|_{\alpha=\alpha^*} = E[(\tilde{z} - (1+\lambda)\bar{z}) \cdot U'(w - \tilde{y} - (1-\alpha^*)\tilde{z} - \alpha^*(1+\lambda)\bar{z})] = 0, \qquad (3)$$

where α^* is the optimal coinsurance rate in the presence of the background risk. The second-order condition for a unique maximum is given by

$$(d^2EU/d\alpha^2)|_{\alpha=\alpha^*} = E[(\tilde{z} - (1+\lambda)\bar{z})^2 \cdot U''(w - \tilde{y} - (1-\alpha^*)\tilde{z} - \alpha^*(1+\lambda)\bar{z})] < 0. \qquad (4)$$

The strict concavity of U guarantees that (4) is satisfied.

Notice that, when $\tilde{y}$ is absent, we are back to the case of complete insurance market. Denote the optimal coinsurance rate in the absence of $\tilde{y}$ by α_0. It is easy to verify from (3) that, $\alpha_0 = (>/<) 1$ if, and only if

[6] In Eeckhoudt and Kimball's [8] paper, the background risk is added to the individual's wealth. However, in this paper, the background risk is subtracted from the individual's wealth. This formulation makes the discussion of the results derived in this section easier.

$\lambda = (< / >)\, 0$. In the absence of $\widetilde{y}$, an individual has "full-insurance coverage" if all risk is eliminated after the purchase of an insurance (in this case when $\alpha = 1$). Clearly, an individual has optimal full- (over-/under-) insurance coverage if, and only if insurance is fair (favorable/ unfavorable) which is a standard result in the literature of complete insurance markets (see, e.g., Eeckhoudt and Kimball [8]).

Under what conditions does an individual have full-insurance coverage in the presence of background risk? Unfortunately, in the presence of background risk, the above definition of "full-insurance coverage" is no longer workable. It is apparent from (1) that it is impossible to eliminate all risk by means of purchasing insurance, except for the special case in which $\widetilde{y}$ and $\widetilde{z}$ are perfectly correlated (see Schlesinger and Doherty (1985)), i.e., $\widetilde{y} = b\widetilde{z}$, almost surely, where b is a constant. A workable and yet reasonable definition that can be employed in the context of incomplete insurance markets is as follows:

Definition: *An individual has "full-insurance coverage on average" if the insurable risk is eliminated when the uninsurable risk is evaluated at its mean.*

In the presence of background risk, "full-insurance coverage on average" requires that the coinsurance rate equals one, which by coincidence coincides with the definition of "full-insurance coverage" in complete markets. To see this, it suffices to check that, when $\alpha = 1$ and $\widetilde{y} = \overline{y}$, the wealth level becomes $w - \overline{y} - (1+\lambda)\overline{z}$ which does not involve $\widetilde{z}$. One should not, however, expect that full-insurance coverage on average is always equivalent to $\alpha = 1$. It will be shown in Section 3 that, in the presence of default risk, full-insurance coverage on average requires $\alpha > 1$.

Before proceeding, let us define a new random variable $\widetilde{\varepsilon}$ which is uncorrelated with the insurable risk. To do that, let constant $b = \mathrm{cov}(\widetilde{y},\widetilde{z})/\mathrm{var}(\widetilde{z})$ and $\rho_{z,y}$ be the correlation coefficient of $\widetilde{z}$ and $\widetilde{y}$. Clearly, b and $\rho_{z,y}$ have the same sign. Construct the new random variable $\widetilde{\varepsilon}$ such that $\widetilde{\varepsilon} = \widetilde{y} - b\widetilde{z}$, almost surely. Rearranging terms yields

$$\widetilde{y} = b\widetilde{z} + \widetilde{\varepsilon}\text{, almost surely.} \tag{5}$$

Clearly, the perfect correlation assumption simply restricts $\widetilde{\varepsilon}$ to be degenerate. Finally, it can be checked that $\mathrm{cov}(\widetilde{z},\widetilde{\varepsilon}) = 0$.[7]

It turns out that if we employ the "regressibility assumption" used in the literature of indirect hedging of exchange rate risk (see, e.g., Broll, Wahl, and Zilcha [2] and Broll and Wahl [3]), the analysis will be substantially simplified.[8] The "regressibility assumption" specifies that there exists $\widetilde{\varepsilon}$ satisfying (5) and $\widetilde{\varepsilon}$ is independent of $\widetilde{z}$.[9]

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[7] To see this, take expectation on both sides of (5) and rearrange to get $\overline{\varepsilon} = \overline{y} - b\overline{z}$, where $\overline{\varepsilon}$ denotes the mean of $\widetilde{\varepsilon}$. This together with (5) implies that $\widetilde{\varepsilon} - \overline{\varepsilon} = (\widetilde{y} - \overline{y}) - b(\widetilde{z} - \overline{z}))$, almost surely. Multiplying both sides by $\widetilde{z} - \overline{z}$ and taking expectation gives the result.

[8] Broll, Wahl, and Zilcha [2] and Broll and Wahl [3] analyze the case in which an exporting firm facing domestic exchange rate risk purchases alternative futures contracts which are not perfectly correlated to the domestic exchange rate when a forward exchange market for domestic exchange does not exist.

[9] Notice that the independence of $\widetilde{\varepsilon}$ and $\widetilde{z}$ implies $\mathrm{cov}(\widetilde{\varepsilon},\widetilde{z}) = 0$ but not vice versa. In fact, the regressibility assumption is a strengthening of the fact that $\mathrm{cov}(\widetilde{\varepsilon},\widetilde{z}) = 0$ by ignoring higher-order joint moments of $\widetilde{\varepsilon}$ and $\widetilde{z}$. Imagine that $\widetilde{z}$ is the insurable medical expenditure risk, $\widetilde{y}$ is the risk of wage income loss, and $\widetilde{\varepsilon}$ is a linear combination of all other factors affecting the risk of wage income loss that is uncorrelated to the risk of medical expenditure. For instance, $\widetilde{\varepsilon}$ may include (mainly) random productivity shocks, random demand for the product that the individual produces, and frictional or structural unemployment, etc. The risk of medical expenditure, on the other hand, depends mainly on the hospital (or doctor) randomly picked, the random price of a certain treatment, and the random health condition of the

Under the regressibility assumption, (5) can be interpreted as the regression of $\tilde{y}$ on $\tilde{z}$ with an independent noise $\tilde{\varepsilon}$ and regression coefficient b. Apart from the consideration of parsimony, the regressibility assumption is employed because it serves as a generalization of the independence assumption, the perfect correlation assumption, and the bivariate normal assumption used by Eeckhoudt and Kimball [8], Schlesinger and Doherty (1985), and Doherty and Schlesinger [6], respectively. Clearly, if $\tilde{y}$ is independent of $\tilde{z}$, then $\tilde{\varepsilon}$ is also independent of $\tilde{z}$ satisfying the regressibility assumption. Finally, the following states that the bivariate normal assumption implies regressibility:

Claim *If $\tilde{y}$ and $\tilde{z}$ have a bivariate normal distribution, then $\tilde{z}$ and $\tilde{\varepsilon}$ are statistically independent satisfying the regressibility assumption.*

Proof: The claim can be inferred from the results in Casella and Berger [4, pp.167-8]. A detailed proof is available from the author upon request. □

The following theorem states the relation between optimal coinsurance rate, loading factor, and the regression coefficient of the insurable risk on the background risk:[10]

Theorem 1 *If the regressibility assumption holds, then $\alpha^* = (> / <) 1+b$ if, and only if $\lambda \doteq (< / >) 0$.*

Proof: Define $\theta = (1+b)^{-1}\alpha$, $\theta^* = (1+b)^{-1}\alpha^*$, $\tilde{\eta} = (1+b)\tilde{z}$, and $\bar{\eta} = (1+b)\bar{z}$. Substituting these into (2) and using (5), the individual can be thought of as choosing θ to maximize

$$EU = EU[w - \tilde{\varepsilon} - (1-\theta)\tilde{\eta} - \theta(1+\lambda)\bar{\eta}]. \quad (6)$$

The regressibility assumption implies that $\tilde{\eta}$ and $\tilde{\varepsilon}$ are independent. Therefore, from Eeckhoudt and Kimball's [8] Proposition 1 for the case of independent background risk, $\lambda = 0$ implies (the optimum of (6)) $\theta^* = 1$ and hence $\alpha^* = 1+b$. We can infer from their Proposition 2 that when $\lambda > (<) 0$, $\theta^* < (>) 1$ and hence $\alpha^* < (>) 1+b$. □

Before explaining the intuition behind the theorem, notice that, the regressibility assumption can be replaced by the "small-risk assumption" (as defined by Samuelson (1970)) together with second-order Taylor approximation.[11][11] Theorem 1 says that, when the regressibility assumption holds, optimal insurance coverage depends crucially on the loading factor λ and the value of b. First, in the presence of an uncorrelated background risk (i.e., $b = 0$), an individual fully insures (over-insures/under-insures) *on average* if, and only if insurance is fair (favorable/ unfavorable). Eeckhoudt and Kimball [8] derive a similar result for the case of independent background risk.

Second, when the background risk is positively correlated to the insurable risk (i.e., $b > 0$), the individual over-insures on average when insurance is fair or favorable. This is rather intuitive as a positive correlation between $\tilde{z}$ and $\tilde{y}$ means that it is more likely that the insurable loss is large when the

individual. In this case, the marginal probabilities and the probabilities of random factors other than medical expenditure risk conditional on medical expenditure risk do not seem to deviate by much. Therefore, the assumption that $\tilde{\varepsilon}$ and $\tilde{z}$ are independent seems appropriate.

[10] I thank a referee for suggesting how I can simplify the proofs of the two theorems in this paper.

[11] There is a full discussion of this claim in the next section. Here, small risks are risks with (joint) small-risk distributions. A family of small-risk or compact distributions is such that, as some specified parameter goes to zero, all the distributions converge to a sure outcome. For a detailed discussion of the definition of a small-risk distribution and the limitations of applying this assumption to the portfolio selection problem, see Samuelson [12].

uninsurable loss is also large. When insurance is fair, the individual over-insures in the insurable risk so as to insure (at least partly) against the uninsurable loss provided that there is no upper bound on insurance coverage. Actuarial favorableness serves to further increase the insurance coverage. Notice that a coinsurance rate of greater-than-one is possible for the case of life insurance but not for the case of health insurance. For the latter kind of insurance, the insured can at most be compensated for the realized value of the insurable loss; therefore, actual coinsurance rate is less than or equal to the optimal coinsurance rate.

Finally, when the background risk is negatively correlated with the insurable risk (i.e., $b < 0$), the individual under-insures on average when insurance is fair or unfavorable. In this case, the uninsurable risk serves to offset (at least partly) the insurable risk. These results generalize the results of Doherty and Schlesinger [5] and Schlesinger and Doherty (1985). However, these results are less general but more intuitive than those of Eeckhoudt and Kimball [8]. Their results involve the concepts of 'third-order conditional stochastic dominance' and 'absolute prudence.'[12]

3. Default Risk and Insurance Coverage[13]

In this section, a much more generalized version of Doherty and Schlesinger's [7] three-state model is formulated and analyzed. Suppose an individual is subject to an insurable risk as before. Assume now that there is some possibility that the insurance company may default; i.e., the individual may not be fully compensated for the covered part of the realized loss. Furthermore, assume that this default risk is not insurable. Denote the compensation ratio by $\tilde{\gamma}$ which is a random variable with realizations $0 \le \gamma \le 1$, mean $\bar{\gamma} > 0$, and standard deviation σ_γ. Assume also that the joint distribution of $\tilde{\gamma}$ and $\tilde{z}$ is known to both the insurer and the insured. Finally, the definitions of w, λ, $\tilde{z}$, and U follow those in the previous section.

When a coinsurance rate α is chosen, the individual has to pay a premium $\alpha(1+\lambda)E(\tilde{\gamma}\,\tilde{z})$ after taking into account the default risk. Let $\tilde{x} = \tilde{\gamma}\,\tilde{z}$ and $\bar{x} = E\tilde{x} = E(\tilde{\gamma}\,\tilde{z})$. Here, $\tilde{x}$ is the random compensation when the coinsurance rate equals one. The individual's random wealth is given by

$$\begin{aligned} & w - \tilde{z} + \alpha\tilde{\gamma}\,\tilde{z} - \alpha(1+\lambda)E(\tilde{\gamma}\,\tilde{z}) \\ & = w - \tilde{z} + \alpha\tilde{x} - \alpha(1+\lambda)\bar{x} \end{aligned} . \tag{7}$$

Her problem is to choose α to maximize her von Neumann-Morgenstern expected utility

$$EU = EU(w - \tilde{z} + \alpha\tilde{x} - \alpha(1+\lambda)\bar{x}). \tag{8}$$

The first-order condition for an optimum is given by

$$\begin{aligned} & (dEU/d\alpha)|_{\alpha=\alpha^{**}} = E[(\tilde{x} - (1+\lambda)\bar{x}) \cdot \\ & U'(w - \tilde{z} + \alpha^{**}\tilde{x} - \alpha^{**}(1+\lambda)\bar{x})] = 0 \end{aligned}, \tag{9}$$

where α^{**} is the optimal coinsurance rate in the presence of default risk. The second-order condition for a unique maximum is given by

$$\begin{aligned} & (d^2EU/d\alpha^2)|_{\alpha=\alpha^{**}} = E[(\tilde{x} - (1+\lambda)\bar{x})^2 \cdot \\ & U''(w - \tilde{z} + \alpha^{**}\tilde{x} - \alpha^{**}(1+\lambda)\bar{x})] < 0. \end{aligned} \tag{10}$$

The strict concavity of U guarantees that (10) is satisfied.

Similar to the previous section, construct a random variable $\tilde{\xi}$ such that $\tilde{\xi} = \tilde{z} - c\tilde{\gamma}\,\tilde{z} = \tilde{z} - c\tilde{x}$, almost surely, where $c = \text{cov}(\tilde{z},\tilde{x})/\text{var}(\tilde{x})$. Rearranging terms yields

[12] Interested readers should consult Eeckhoudt and Kimball's [8] paper for the exact definitions of these concepts. Notice that the case of a negative correlation between the background and the insurable risks in this paper corresponds to the case of a positive correlation in their paper. This difference is due to the fact that the background risk is treated as a random loss in this paper but a random gain in their paper.

[13] As suggested by a referee, another name for 'default risk' is the 'risk of partial payment.' The model in this section covers all the cases under which there exists some possibility that the insurance company may not fully compensate for the insured losses of the insured.

$$\tilde{z} = c\tilde{x} + \tilde{\xi}\text{, almost surely.} \quad (11)$$

It is easy to verify that $\text{cov}(\tilde{\xi}, \tilde{x}) = 0$. Under the regressibility assumption, $\tilde{\xi}$ is assumed to be statistically independent of $\tilde{x}$. (11) can be interpreted as the regression of $\tilde{z}$ on $\tilde{x}$ with independent noise $\tilde{\xi}$ and regression coefficient c.

The following theorem shows that the case of default risk is somehow different from the case of background risk. Such a difference is due to the fact that, while the background risk is additive to the insurable risk, the default risk is multiplicative to the insurable risk.

Theorem 2 *If the regressibility assumption holds, then* $\alpha^{**} = (> / <)\, c$ *if, and only if* $\lambda = (< / >)\, 0$.

Proof: Redefine $\theta = \alpha / c$, $\theta^* = \alpha^{**}/c$, $\tilde{\eta} = c\tilde{x}$, and $\overline{\eta} = c\overline{x}$. Substituting these into (8), using (11), and setting $\tilde{\xi} = \tilde{\varepsilon}$, the individual's maximization problem can be thought of as choosing θ to maximize (6). The regressibility assumption allows us to infer from Eeckhoudt and Kimball's [8] Proposition 1 that when $\lambda = 0$, $\theta^* = 1$ and hence $\alpha^{**} = c$. We can infer from their Proposition 2 that when $\lambda > (<)\, 0$, $\theta^* < (>)\, 1$ and hence $\alpha^{**} < (>)\, c$. □

Before explaining the implication of Theorem 2, notice that, similar to Theorem 1, the regressibility assumption can be replaced by the "small-risk assumption" as a sufficient condition.[14] To understand the implication of Theorem 2, it suffices to focus on the value $c = \text{cov}(\tilde{z}, \tilde{x})/\text{var}(\tilde{x})$. Consider first the benchmark case in which the default risk is degenerate, i.e., $\tilde{\gamma} = \overline{\gamma}$, almost surely. It is easy to check that

$$c = \frac{\text{cov}(\overline{\gamma}\,\tilde{z}, \tilde{z})}{\text{var}(\overline{\gamma}\,\tilde{z})} = \frac{\overline{\gamma}\sigma_z^{\,2}}{\overline{\gamma}^2\sigma_z^{\,2}} = \frac{1}{\overline{\gamma}}. \quad (12)$$

Equation (12) together with Theorem 2 implies that $\alpha^{**} = (> / <)\, 1/\overline{\gamma}$ if, and only if, $\lambda = (< / >)\, 0$. The individual has full-insurance on average if, and only if insurance is fair while over-insurance and under-insurance on average are associated with favorable and unfavorable insurance, respectively. To see this, substitute $\tilde{\gamma} = \overline{\gamma}$ into the L.H.S. of (7) to get

$$w - (1 - \alpha\overline{\gamma})\tilde{z} - \alpha(1+\lambda)\overline{\gamma}\,\overline{z}. \quad (13)$$

Equation (13) implies that $\tilde{z}$ is eliminated if, and only if $\alpha = 1/\overline{\gamma}$. Therefore, full insurance on average is equivalent to $\alpha = 1/\overline{\gamma}$. Notice that full insurance on average in the presence of certain default is larger than that in the absence of any default whenever $\overline{\gamma} < 1$. The

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[14] The following shows how the small-risk assumption replaces the regressibility assumption. The first-order condition of (6) is given by

$$E[(\tilde{\eta} - (1+\lambda)\overline{\eta}) \cdot U'(w - \tilde{\varepsilon} - (1-\theta^*)\tilde{\eta} - \theta^*(1+\lambda)\overline{\eta}] = 0.$$

The L.H.S. can be written as

$$\text{cov}(\tilde{\eta}, U'(w - \tilde{\varepsilon} - (1-\theta^*)\tilde{\eta} - \theta(1+\lambda)\overline{\eta})) - \lambda\overline{\eta}EU'.$$

Clearly, the sign of the first term is the same as that of λ. When the "small-risk assumption" holds, second-order Taylor series expansion of the first term at $(\overline{\eta}, \overline{\varepsilon})$ gives $-[\text{cov}(\tilde{\eta}, \tilde{\varepsilon}) + (1-\theta^*)\text{var}(\tilde{\eta})] \cdot U''$. Since $\text{cov}(\tilde{\eta}, \tilde{\varepsilon}) = \text{cov}(\tilde{z}, \tilde{\varepsilon}) = 0$ by construction. The sign of the first term, which is the same as that of λ, is the same as that of $1 - \theta$ yielding the same result as shown in the proofs of the Theorems. Notice that while a real-world example for the sufficient condition of regressibility to hold is not easy to construct for the case of default risk, we can still justify the results in Theorem 2 by invoking the "small-risk assumption." In fact, the second-order Taylor expansion shows that the above results hold whenever the third and higher joint-moments of $\tilde{z}$ and $\tilde{\varepsilon}$ or the third- and higher-derivatives of U are sufficiently small. The former condition is met whenever the default and the insurable risks have sufficiently low probabilities for 'extreme' realizations.

presence of certain default requires the individual to raise her insurance coverage by $1/\bar{y}$ times so as to eliminate the insurable risk.

Next, consider the case in which $\tilde{y}$ is non-degenerate and is independent of $\tilde{z}$. It can be verified that $\text{cov}(\tilde{y}\,\tilde{z},\tilde{z}) = \bar{y}\,\sigma_z^{\,2}$ and $\text{var}(\tilde{y}\,\tilde{z}) = \text{E}(\tilde{y}^2)\sigma_z^{\,2} + \sigma_y^{\,2}E(\tilde{z}^2)$. Therefore,

$$c = \frac{\bar{y}\,\sigma_z^{\,2}}{E(\tilde{y}^2\sigma_z^{\,2} + \sigma_y^{\,2}E(\tilde{z}^2))} < \frac{\bar{y}\sigma_z^{\,2}}{\text{E}(\tilde{y}^2)\sigma_z^{\,2}} < \frac{1}{\bar{y}}. \tag{14}$$

(14) together with Theorem 2 implies that $\alpha^{**} < 1/\bar{y}$ when $\lambda \geq 0$. In other words, under-insurance on average is optimal when insurance is fair or unfavorable. This result together with that of the benchmark case suggests that the effect of an independent default risk on insurance coverage can be broken down into two parts:

1. The increase in insurance coverage due to the decrease in average compensation.
2. The decrease in insurance coverage due to the additional risk introduced by the uncertainty of the compensation rate.

The above decomposition explains the ambiguous results obtained from Doherty and Schlesinger's [7] simple three-state model. That is, in the presence of an independent default risk, optimal coinsurance rate may be one, less-than-one or greater-than-one even when insurance is fair. With fair insurance, an average compensation rate of less than one tends to push up the coinsurance rate (towards $1/\bar{y}$). The uncertainty of the compensation rate, on the other hand, tends to pull down the coinsurance rate (below $1/\bar{y}$) as a higher insurance coverage raises the default risk faced by the individual. Consequently, the optimal coinsurance rate can be larger or smaller than one depending on the relative strength of the two opposing forces.

Finally, suppose $\tilde{y}$ and $\tilde{z}$ are not statistically independent. Applying second-order Taylor expansion around $\bar{y}$ and $\bar{z}$ yields

$$\text{cov}(\tilde{y}\,\tilde{z},\tilde{z}) \approx \bar{z}\text{cov}(\tilde{y},\tilde{z}) + \bar{y}\,\text{var}(\tilde{z}); \tag{15}$$

$$\text{var}(\tilde{y}\,\tilde{z}) \approx \bar{y}^2\text{var}(\tilde{z}) + \bar{z}^2\text{var}(\tilde{y}) + 2\bar{y}\,\bar{z}\,\text{cov}(\tilde{y},\tilde{z}) - [\text{cov}(\tilde{y},\tilde{z})]^2. \tag{16}$$

Let $k_{y,z} = \sigma_y/\sigma_z$ and $\rho_{y,z}$ be the correlation coefficient of $\tilde{y}$ and $\tilde{z}$. Furthermore, let

$$\rho^- = \frac{-2\bar{z}k_{y,z}}{\sqrt{\bar{y}^2 + 4\sigma_y^{\,2}} + \bar{y}^2} < 0 \text{ and}$$

$$\rho^+ = \frac{2\bar{z}k_{y,z}}{\sqrt{\bar{y}^2 + 4\sigma_y^{\,2}} - \bar{y}^2} < 0.$$

It can be verified from (15) and (16) that, subject to $-1 \leq \rho_{y,z} \leq 1$, the following (approximate) relations hold:[15]

$$\begin{cases} \text{(i) } c < 1/\bar{y} \text{ iff } \rho_{y,z} \in (\rho^-,\rho^+) \\ \text{(ii) } c = 1/\bar{y} \text{ iff } \rho_{y,z} = \rho^- \text{ or } \rho_{y,z} = \rho^+ \\ \text{(iii) } c > 1/\bar{y} \text{ iff } \rho_{y,z} \notin [\rho^-,\rho^+] \end{cases}$$

These relations together with Theorem 2 imply that, when the default risk and the insurable risk are uncorrelated (i.e., when $\rho_{y,z} = 0$), optimal coinsurance rate is less than $1/\bar{y}$ and hence under-insurance on average is optimal when insurance is fair or unfavorable. This reinforces the conclusion derived for the case when $\tilde{y}$ and $\tilde{z}$ are independent. Another observation is that merely knowing the sign of $\rho_{y,z}$ is not enough for determining whether full- (under-/over-) insurance on average is optimal even when insurance is fair. This result contradicts that of the case of uninsurable background risk in which a positive (negative) correlation with the insurable risk tends to raise (reduce) insurance coverage.

[15] A detailed proof of this result is available from the author upon request.

4. Concluding Remarks

This paper has compared an individual's optimal insurance coverage under two types of market incompleteness, namely, the presence of uninsurable background risk and the presence of default risk. In both cases, under the regressibility assumption, it is found that optimal insurance coverage is affected by actuarial unfairness as well as the correlation between the insurable and the uninsurable risks. With fair insurance, while full insurance on average is optimal for the case of uncorrelated background risk, under-insurance on average is optimal for the case of independent default risk. The main reason for this difference is that, unlike the background risk which is additive to the insurable risk, the default risk is multiplied to it. The multiplicative nature of the independent default risk tends to suppress insurance coverage as the 'total' risk faced by the individual gets larger when more insurance is purchased. Finally, it is found that the sign of the correlation between the idiosyncratic and the insurable risks determines whether over- or under-insurance on average is optimal when insurance is fair. On the other hand, the multiplicative nature of the default risk renders the relation between its correlation with the insurable risk and the optimal coinsurance rate much more complicated. A positive (negative) correlation between the default risk and the insurable risk does not necessarily imply over-insurance (under-insurance) on average when insurance is fair.

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