

alternative robust approach described in this paper.

We employ a slider-crank mechanism as the design example, which includes random design variables and nonlinear design constraints. Both methods of maximizing the goal-reliability and fuzzy formulation are investigated under the same design conditions. The fuzzy formulation is extended to deal with a robust multiple-goal structural problem under three loading conditions and with uncertainties. That example includes random design variables and fuzzy probabilistic constraints in stead of conventional constraints. These assumptions assure that the designer can control the design safety. The mathematical formulations of the above strategies, solution procedure and the numerical results are presented, discussed and compared in the paper consequently.

## II. RELIABILITY INDICATOR OF A DESIGN GOAL

Since many uncertainties are transformed into standard, uncorrelated and normal distribution, therefore in this paper we only consider the random variables or parameters with normal distribution. For non-normal dependent random variables, they can be transformed into the standardized normal variables using Rosenblatt transformation [9]. The form of a reliability indicator,  $R_F$ , corresponding to a design goal (objective or target) function  $F$  is written as follows:

$$R_F = \Phi^{-1} \left\{ \frac{C_{F(X)} - E[F(X)]}{\sigma_{F(X)}} \right\} \quad (1)$$

where  $\Phi^{-1}$  is the inverse standard normal cumulative distribution function;  $C_{F(X)}$  is the reference point for the reliability of goal-performance, and  $\sigma_{F(X)}$  is the standard deviation of design performance. For the computation convenience, sometimes  $\sigma_{F(X)}$  can be replaced by the variance of  $F(X)$  written as  $\text{Var}[F(X)]$ . The value of  $F(X)$  is obtained by substituting the mean value of  $X$  for this goal function. The standard deviation  $\sigma_{F(X)}$  of the function  $F(X)$  is obtained by the following:

$$\sigma_{F(X)} = \sqrt{S_{F(X)}^2} \quad (2)$$

The estimated variance of the goal function,  $S_{F(X)}^2$ , is simply obtained by the  $N = 2^I$  full factorial design experiment [3], where the superscript "I" shows the number of variables with the variation.  $S_{F(X)}$  is expressed as the following form:

$$S_{F(X)}^2 = \frac{1}{N} \sum_{n=1}^N (F^{(n)}(X) - F)^2 \quad (3)$$

where

$$F = \frac{1}{N} \sum_{n=1}^N F^{(n)}(X) \quad (4)$$

In Eq. (4),  $F$  represents the average value of  $F^{(n)}(X)$  which is the value obtained from substituting the  $n = 1$  to  $N$  combinations for  $F(X)$ . The expectation of goal function,  $E[F(X)]$ , is simply derived from the partial derivative or Taylor series expansion method [13] with a quadratic term.

$$E[F(X)] = F(X)|_X + \frac{1}{2} \sum_{k=1}^n \frac{\partial^2 F(X)}{\partial^2 X_k^2} \text{Var}(X_k) \quad (5)$$

The formulation of  $C_{F(X)}$  is defined as the following equation:

$$C_{F(X)} = F^*(X) \pm \alpha \sigma_{F(X)}^* \quad (6)$$

where  $F^*(X)$  is the optimum mean value by the optimization procedure.  $\sigma_{F(X)}^*$  is the minimum value of the standard deviation also obtained from optimization procedure.  $\alpha$  is a parameter that we can tune depending on the mean value so that  $R_{F(X)}$  will not be too large (unity) or too small (zero). We use the plus sign in Eq. (6) where the goal function is to be minimized, and the minus sign for maximizing the goal function. The situation for the minimization of a design target is depicted in Fig. 1. For the cases when a goal is to be maximized or minimized, there are two forms of the reliability indicator. Eq. (1) is the form of the reliability indicator that minimizes a goal-function's mean value. The form of a reliability indicator that maximizes a goal-function's mean value is written as follows:

$$R_F = \Phi^{-1} \left\{ \frac{E[F(X)] - C_{F(X)}}{\sigma_{F(X)}} \right\} \quad (7)$$

Since the  $C_F$  value has a predetermined value (Eq. 6), from Eq. (1) and Eq. (7) that the maximization of the reliability indicator is equivalent to optimize (minimize or maximize) the goal-function and minimize its standard deviation simultaneously. If  $C_F$  has an expected value, the resultant reliability indicator is an actual goal reliability of that design system.

$$\begin{aligned} & \text{Maximize } \lambda \\ & \text{subject to} \end{aligned} \quad (11)$$

$$\lambda - \mu_{F(X)} \leq 0 \quad (12)$$

$$\lambda - \mu_{SF(X)} \leq 0 \quad (13)$$

$$\lambda - \mu_{g_i(X)} \leq 0 \quad i = 1, 2, \dots, m \quad (14)$$

where  $\lambda$  is a scalar;  $\mu_{F(X)}$  and  $\mu_{SF(X)}$  are the membership functions corresponding to the goal function and its variance, respectively. For detailed construction of these membership functions one can refer to the related references [7,8,10].

### 3. Robust Multiobjective Optimum Design with Fuzzy Probabilistic Constraints

Sometimes a designer has to deal with not only more than two design goals but also a constraint as a whole has a certain expectation of occurrence value. This expectation of occurrence can be a deterministic value or a fuzzy number. The general mathematical formulation for such a robust stochastic problem can be written as minimize

$$[f_{1F}(X), f_{1\sigma}(X), f_{2F}(X), f_{2\sigma}(X), \dots, f_{kF}(X), f_{k\sigma}(X)]^T,$$

where  $f_{iF}(X)$  is the  $i$ th goal-function of expectation and  $f_{i\sigma}(X)$  is the  $i$ th goal-function of variance. The optimization problem can be subject to the following two types of constraints:

- (1)  $P(g_i(X) \leq 0) \geq P_{i,\text{exp}}$  where  $P_{i,\text{exp}}$  has a deterministic value.
- (2)  $P(g_i(X) \leq 0) \geq P_{i,\text{fuzzy}}$  where  $P_{i,\text{fuzzy}}$  is a fuzzy number.

The first step is to solve each single goal optimization under the strict environment and calculate the corresponding value of the other goal functions. Then one repeats this procedure again, this time only the allowable fuzzy tolerance of design constraints is relaxed. Choosing the maximum and minimum values of the goal functions among these results, one can construct the appropriate membership functions of the goal functions. Applying the method of fuzzy optimization, one can get the solution of this robust multiple goals design. The paper by Shih and Wangsawidjaja [11] has a better illustration for this type optimization problem. A mechanical structure borrowed and modified from the reference of Chang [2] illustrates the above robust design methods. The mathematical formulation of design optimization and a step-by-step

Fig1. The reliability index and functional distribution at the optimum target by minimization.

## III. ROBUST DESIGN OPTIMIZATION

A robust optimum design of a single goal-function can be done by two different proposed strategies. The first is to directly maximize the reliability indicator introduced in the previous section. The second strategy can be achieved by fuzzy optimization that optimizes the goal function and simultaneously minimizes its standard deviation (or variance).

### 1. Method of Maximization of Reliability Indicator

For the first robust design of optimization problem, the mathematical form is written as follows:

$$\text{Maximize } R_{F(X)} \quad (8)$$

subject to

$$g_i(X) \leq C_i, \quad i = 1, 2, \dots, m \quad (9)$$

$$h_j(X) = C_j, \quad j = m + 1, \dots, M \quad (10)$$

where the form of  $R_{F(X)}$  is expressed in Eq. (1) or Eq. (7). Functions of  $g_i(X)$  and  $h_j(X)$  represent the  $i$ th inequality and the  $j$ th equality constraints, respectively that are in terms of the design variable vector  $X$ .

### 2. Method of Fuzzy Optimization

The second alternative robust method of maximizing reliability has the following mathematical form:

solution process are described in the next section.

#### IV. ILLUSTRATIVE DESIGN EXAMPLES

The kinematic structural diagram of a slider-crank is shown in Fig. 2. The travel distance of the piston,  $F(X)$ , must be maximized where the given angle  $\theta$  is  $15^\circ$ . The design variables are the length of the two cranks depicted as  $x_1$  and  $x_2$ . To start solving this robust design problem, we must first find out the reference value of  $C_{F(X)}$ . The explanation of how to decide this value is given in above section. To obtain optimum  $F^*(X)$  in Eq. (6), we make an optimization formulation as follows:

$$\text{Maximize } E[F(X)] = F(X) + \frac{1}{2} \sum_{i=1}^2 \frac{\partial F^2(X)}{\partial X_i^2} \text{Var}(X_i) \quad (15)$$

$$F(X) = x_1 + x_2 - x_1 \cos \theta - (x_2^2 - x_1^2 \sin^2 \theta)^{\frac{1}{2}} \quad (16)$$

where  $\text{Var}(X_i) = 0.1$  ( $i=1,2$ ) and subject to the following constraints:

$$[(x_1 + 0.1)^2 \sin^2 15^\circ + 0.05] / (x_2 - 0.1)^2 - 1 \leq 0 \quad (17)$$

$$1.0 \leq x_i \leq 10.0 \quad i = 1, 2 \quad (18)$$

Completing the above optimization problem we obtain  $F^*(X) = E[F(X)] = 2.2398$  and  $X = 10., 2.7187$ . The corresponding  $\sigma_{F(X)}$  is 0.3184. Next, we apply factorial design to compute the numerical experiment that the variation form as Eq. (3) is the goal function that has to be minimized. The design constraints still are (17) and (18). The final  $\sigma_{F(X)}^*$  is 0.0040778 and  $X = 1.0, 10.$ . The corresponding  $E[F(X)]$  is 0.056957. With the substitution of  $\alpha = 158$  for Eq. (6) the value of  $C_{F(X)}$  is 1.5955.

##### 1. Reliability-Based Optimization

We calculate Eq. (2) and substitute Eq. (15) for Eq. (1) that has to be maximized with  $C_{F(X)} = 1.5955$ . It is obvious to see that the maximization of goal reliability of  $R_F$  is equivalent to the maximization of  $E[F(X)]$  and the minimization of  $\sigma_{F(X)}$  simultaneously. The design constraints are Eq. (17) and (18) plus the followings:

$$C_{F(X)} / E[F(X)] - 1 \leq 0. \quad (19)$$

$$0.0040778 / \sigma_{F(X)}^* - 1 \leq 0. \quad (20)$$

The optimum solutions are of  $R_F = 0.996$ ,  $E[F(X)] = 2.0364$  and  $\sigma_{F(X)} = 0.16639$  in which  $x_1 = 10.$  and  $x_2 = 2.8324$ .

Fig2. A slider-crank mechanism.

##### 2. Fuzzy Optimization

By applying the second approach to solve this problem, we adopt the linear membership functions with fuzzy mathematical formulation of Eq. (11) to Eq. (14). The optimum solutions are of  $R_F = 0.993$ ,  $E[F(X)] = 1.9022$  and  $\sigma_{F(X)} = 0.12524$  in which  $x_1 = 10.$  and  $x_2 = 2.94$ . For a better observation, we summarize the optimum results of the slider-crank design in Table 1.

From those two kinds of approaches we found out that the value of  $R_F$  in the fuzzy approach is a little lower. This may be due to the numerical errors of large computation in fuzzy optimization. It may be also due to a compromise between the minimum mean and minimum variance value of the goal function that needs to be reached. However, fuzzy approach provides another way to solve the multi-goals optimization problem and/or a problem containing fuzzy information. The little numerical difference will disappear if one

solves the problem with a precision technique.

In order to examine the situation of a reliability indicator corresponding to the variation, we made a small investigation. Table 2 shows a series of results obtained by maximizing  $R_{F(X)}$  with a fixed target of  $E[F(X)]$ . We found out that the variance of a goal function is proportional to its expected mean value. Therefore, when the mean value changes drastically, the variation also has a drastic change. The maximum goal-reliability does not exist at the smallest goal variation. It happens at a certain optimum situation. Therefore, the optimum goal reliability obtained by a predetermined fixed target performance is not better than that by optimizing this target performance simultaneously. In our case, one can see the highest goal reliability (0.9959) appears in italic font obtained in this paper. In actual design, it also is difficult to set a predetermined fixed target because the design space has many feasible solutions.

### 3. Robust Multiobjective Optimum Design with Fuzzy Probabilistic Constraints

The design of a three-bar truss is frequently used in describing an optimization method. This asymmetric three-bar truss for multiple performances shown in Fig. 3 is borrowed from Arora's book [1], but here the goals are simultaneously to optimize the structural weight,

maximum deflection, and the goal-reliability for both weight and deflection. The three cross-sectional areas  $A_1$ ,  $A_2$  and  $A_3$  are design variables given as  $x_1$ ,  $x_2$  and  $x_3$ , respectively. Each variable is normally random with  $x_i \pm 0.1 x_i$ . The stress constraint as a whole has a fuzzy expectation value between 0.98 and 0.999. It means that the satisfying degree of the associated constraint is zero when the computed expectation is less than 0.98 and a complete satisfaction is achieved when that is over 0.999.

The mathematical formulation of this problem is simultaneously to minimize structural weight  $W$ , its variance  $W_\sigma$ , and average deflection  $D$  of three loading and its variation  $D_\sigma$ .

$$W = \gamma \ell (\sqrt{2}A_1 + A_2 + \sqrt{2}A_3) \quad (21)$$

$$W_\sigma = (\sum_{i=1}^8 (W_i - W)^2) / 8 \quad (22)$$

$$D = (\sum_{j=1}^3 \sqrt{u^2 + v^2}) / 3 \quad (23)$$

$$D_\sigma = (\sum_{i=1}^8 (D_i - D)^2) / 8 \quad (24)$$

Table 1 The optimum results for slider-crank design.

	$x_1$	$x_2$	$E[F(X)]$	$\sigma_{F(X)}$	$R_F$
Maximize $E[F(X)]$	10.	2.7187	2.2398	0.31843	0.97846
Minimize $\text{Var}[F(X)]$	1.0	10.	0.0569	0.00408	0.
Reliability-Based design	10.	2.8324	2.0364	0.16639	0.99598
Fuzzy Optimization	10.	2.9402	1.9022	0.12524	0.99284

Table 2 Results of maximizing the reliability index with different fixed target values.

Fixed $E[F(X)]$	1.8	1.9	2.0364	2.1	2.2
$\Phi^{-1}[\ ]$	2.1149	2.5119	2.6500	2.6211	2.3201
$R_{F(X)}$	0.9826	0.9939	0.9959	0.9956	0.9898
$\sigma_{F(X)}$	0.1012	0.1228	0.1669	0.1883	0.2604
$x_1$	10	10	10	10	10
$x_2$	3.02	2.92	2.83	2.79	2.74

The design constraints are:

$$P(\sigma_i(X) - \sigma_{ia} \leq 0) \geq 0.98 \sim 0.99, \quad i = 1, 2, 3 \quad (25)$$

$$-F_i \leq \pi^2 EI / \ell_i^2 \quad i = 1, 2, 3 \quad (26)$$

$$\omega(X) \geq 2500Hz \quad (27)$$

The lower and upper bound of each design variable is

0.65 and 645. cm<sup>2</sup> (0.1 and 100. in<sup>2</sup>), respectively. Because three loading conditions have to be considered together, there are nineteen design constraints.

We formulated the membership functions first to solve this problem by fuzzy formulation. Consequently, the total constraints for maximizing  $\lambda$ -formulation have twenty-three constraints for three loadings. The solutions are  $x_1=157.38$  cm<sup>2</sup> (24.394 in<sup>2</sup>),  $x_2=80.7$  cm<sup>2</sup> (12.509 in<sup>2</sup>),  $x_3=146.47$  cm<sup>2</sup> (22.703 in<sup>2</sup>), with  $C_w=84.445$ ,  $\alpha_w=150.$ ,  $C_D=1.4241E-03$  and  $\alpha_D=5.0E6$ . The resultant reliability of structural weight and the average deflection equals 0.9854255 and 0.960585, respectively. This result shows that both design goals have the minimum variability without reducing the design variables tolerance. The final design has the maximum robust strength for both goals and their reliability.

$\Theta$ (degree)	45	90	135
Load, P(N)	177920	133440	88960

Fig.3 An asymmetric three-bar structure with three loading conditions.

## V. CLOSING REMARKS

A simple computing method to achieve the robust and reliable structural design is proposed by establishing a reliability indicator as the reference. This reliability indicator can be applied to either minimization or maximization of a design goal. To obtain the goal variation, the factorial design of experiment serves as a successful vehicle. Two robust

design strategies of obtaining the maximum goal-reliability of the reference and the optimum expectation of the design goal are presented. For a single goal optimization problem, maximization of the reliability indicator is recommended because of programming simplicity. However this strategy faces a problem that has to simultaneously optimize multiple goals and when the fuzzy information is in the constraints. The presented fuzzy formulation gives a solution to this problem. A mechanical structure of slider-crank is solved to illustrate and verify the proposed methods. Several final designs of fixing design target performance are compared with the result of optimizing the target performance. It is shown that the highest goal-reliability occurs at where the target performance and the performance variation are simultaneously optimized. A multiobjective optimum design of an asymmetric three-bar structure with multiple performances requirements and fuzzy probabilistic constraints is presented to illustrate the proposed methodology of fuzzy formulation.

## ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support of this research by the National Science Council (R.O.C.) under the Grant NSC 87-2212-E-032-007.

## REFERENCES

1. Arora, J. S., *Introduction to Optimum Design*, McGraw-Hill, New York, pp.489-493 (1989).
2. Chang, H., "A Simple Computer-Aided Design Procedure for Minimal Variations," *Computer Methods in Applied Mechanics and Engineering*, Vol. 73, pp. 99-107 (1989).
3. Charnes, A. and Cooper, W. W., "Chance Constrained Programming," *Management & Science*, Vol. 6, pp. 73-79 (1959).
4. Oh, H. L., "Variation Tolerance Design," *Proc. of ASME Winter Annual Meeting, Production Engineering Division*, Vol. 115, pp. 137-146 (1987).
5. Parkinson, A., Sorensen, C. and Pourhassan, N., "A General Approach for Robust Optimal Design," *ASME Journal of Mechanical Design*, Vol. 115, pp. 74-80 (1993).
6. Rao, S. S., "Structural Optimization by Chanced Programming Techniques," *Computers & Structures*, Vol. 12, pp. 777-781 (1980).
7. Rao, S. S., "Multi-Objective Optimization of Fuzzy Structural Systems," *Int. J. Numerical Methods in Eng.*, Vol. 24, pp. 1157-1171 (1987).
8. Rao, S. S., Sundararaju, K., Prakash, B. G. and Balakrishna, C., "Multiobjective Fuzzy Optimization Techniques for Engineering Design," *Computers & Structures*, Vol. 42(1), pp. 37-44 (1992).
9. Rosenblatt, M., "Remarks on a Multivariate Transformation," *Ann. Math. Statist.*, Vol. 23(3), pp. 470-472

- (1952).
10. Shih, C. J. and Lai, T. K., "Fuzzy Weighting Optimization with Several Objective Functions in Structural Design," *Computers & Structures*, Vol. 52(5), pp. 917-924 (1994).
  11. Shih, C. J. and Wangsawidjaja, R. A. S., "Mixed Fuzzy-Probabilistic Programming Approach for Multiobjective Engineering Optimization with Random Variables," *Computers & Structures*, Vol. 59(2), pp. 283-290 (1996).
  12. Torng, T. Y. and Yang, R. J., "A Reliability-Based Design Optimization Procedure for Minimal Variations," *AIAA paper*, No. 94-1415-CP, pp. 842-852 (1994).
  13. Wong, F. S., "First-Order, Second-Moment Methods," *Computers & Structures*, Vol. 20(4), pp. 779-791 (1985).
  14. Zadeh, L., "Fuzzy Sets," *Information and Control*, Vol. 8, pp. 338-353 (1965).

***Manuscript Received: Jan. 20, 1998***

***Revision Received: Mar. 20, 1998***

***and Accepted: Mar. 30, 1998***