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A NOTE ON "GENERALIZED ITERATION PROCESS" BY T. HU AND G.-S. YANG

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In [1], T. Hu and G.-S. Yang obtained the following

THEOREM. *Suppose f is a continuous mapping which maps the closed interval* $[0,1]$ *into itself, and $A=(a_{nk})$ is a stable iteration matrix, then for any $x_1\in[0,1]$, the generalized iteration sequence $\{v_n\}$ converges to a fixed point of f on* $[0,1]$.

Here, a stable iteration matrix $A=(a_{nk})$ is an infinite lower triangular matrix such that

(1) $$a_{nk}\geq 0,\ \sum_{k=1}^{n} a_{nk}=1,$$

(2) $$\lim_{n\to\infty} a_{nn}=0,\ \lim_{n\to\infty} a_{nk}=0,\ k=1,2,\cdots, \text{ and}$$

(3) $$a_{n+1\cdot k}=(1-a_{n+1\cdot n+1})a_{nk} \text{ for } k=1,2,\cdots,n$$

and $\{x_n\}$ and $\{v_n\}$ are defined inductively by

$$v_n=\sum_{k=1}^{n} a_{nk}x_k,\ x_{n+1}=fv_n,\ n=1,2,\cdots \quad [2].$$

In this note, Theorem of Hu and Yang is actually a simple consequence of the following

PROPOSITION ([3], Corollary 3.1). *Let f be a continuous selfmap of a compact interval I, and $\{x_n\}$ a sequence in I such that $v_{n+1}\in\overline{v_n(fv_n)}$ for all $n\in\omega$. Then*
(1) *$v_n-v_{n+1}\to 0$ iff $\{v_n\}$ converges, and*
(2) *$v_n-fv_n\to 0$ iff $\{v_n\}$ converges to a fixed point of f.*

Here, — denotes the closed interval joining two points.
Note that the stable iteration matrix $A=(a_{nk})$ in Theorem is regular. Hence, A maps every convergent sequence into a convergent sequence with invariant limit in the sense that if $x_k\to l$, then $\sum_{k=1}^{\infty} a_{nk}x_k\to l$ (cf. [4]).

Proof of Theorem. Since $|v_{n+1}-v_n|\leq a_{n+1,n+1}\to 0$, $\{v_n\}$ converges to some $v_0\in[0,1]$ by Proposition (1). Therefore, $x_k=f(v_{k-1})\to fv_0$ from the continuity of f. Since A is regular, $v_n=\sum_{k=1}^{n} a_{nk}x_k\to fv_0$. Hence, $v_0=fv_0$.

References

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