2-colouring $\{C_3, C_4\}$-designs

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A design (V, B) with point set V and collection of blocks B is said to have a blocking set S if there exists a non-empty proper subset S of V such that every block in B meets S, and no block in B lies entirely in S. Equivalently, we may colour the points in V with two colours, say red if a point lies in S and blue if it lies in $V \setminus S$, so that no block is monochromatic. If this is possible, the design is said to be 2-colourable. It is well-known that no Steiner triple system (of order greater than 3) can be 2-coloured. A very recent paper on 2-colouring cycle systems, which includes many useful references, is [2].

For the purpose of this note, we may consider an *m-cycle system* of order n to be an edge-disjoint decomposition of K_n into cycles of length m. If a cycle of length m has edges $\{a_i, a_{i+1}\}$ for $1 \leq i \leq m-1$ and $\{a_1, a_m\}$, then we write the cycle as $(a_1, a_2, \ldots, a_m)$ or $(a_1, a_m, a_{m-1}, \ldots, a_2)$ or any cyclic shift of these. A $\{C_3, C_4\}$-design of order n is then an edge-disjoint decomposition of K_n into copies of C_3 and C_4. It is well-known (see [1]) that a $\{C_3, C_4\}$-design of order n with p copies of C_3 and q copies of C_4 exists if and only if n is odd and $3p + 4q = \binom{n}{2}$. In fact it is easy to verify that if $n \equiv i \pmod 8$ where i is odd, then $p = 4t + (i-1)/2$, for

$t = 0, 1, \ldots, \lfloor \frac{n(n-1)}{24} \rfloor$. We shall say that such a $\{C_3, C_4\}$-design is of type (p, q) if it contains p copies of C_3 and q copies of C_4.

Our aim in this note is to prove the following.

Theorem 1 *Let n be odd. Then there exists a $\{C_3, C_4\}$-design of order n and type (p, q) which can be 2-coloured if and only if $3p + 4q = \binom{n}{2}$ and $p \leq (n-1)/2$.*

Proof: Certainly the requirement that $3p + 4q = \binom{n}{2}$ is necessary for *any* $\{C_3, C_4\}$-design of type (p, q). So we start by showing the necessity of $p \leq (n-1)/2$ in a 2-coloured $\{C_3, C_4\}$-design of type (p, q).

Suppose we have a blocking set S of cardinality s in a $\{C_3, C_4\}$-design of order n and type (p, q). Then certainly n must be odd, and counting the number of edges,

$$3p + 4q = \frac{n(n-1)}{2}.$$

Now counting the $s(n-s)$ edges of K_n that join S with the other $n - s$ vertices, each C_3 must have two edges in this set, while each C_4 has either two or four edges in this set. Thus we have

$$2p + 2q \leq s(n-s).$$

These imply that

$$p \leq 2s(n-s) - \frac{n(n-1)}{2}.$$

But certainly $s(n-s) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, and since n is odd, this becomes $s(n-s) \leq \frac{n^2-1}{4}$.

Hence we obtain $p \leq \frac{n-1}{2}$.

Next we construct appropriate designs and give their blocking sets.

First consider cases smaller than $n = 9$. If $n = 3$, the trivial design of type $(1, 0)$ can be 2-coloured. When $n = 5$, the set $\{1, 2\}$ is a blocking set for the design of type $(2, 1)$ with cycles

$$\{(1, 3, 5), (1, 2, 4), (2, 3, 4, 5)\}.$$

When $n = 7$, the only type (p, q) with $p \leq 3$ is $(3, 3)$. The set $\{1, 2, 3\}$ is a blocking set for the following design of type $(3, 3)$:

$$\{(1, 4, 7), (2, 5, 7), (3, 6, 7), (1, 2, 4, 5), (1, 3, 4, 6), (2, 3, 5, 6)\}.$$

Now let $n = 2m + 1$ with $n \geq 9$. We shall construct a $\{C_3, C_4\}$-design of type $(p, q) = (m - 4k, m(m-1)/2 + 3k)$ for each $4k \leq m$. Let the vertex set be

$$\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq m,\ j = 1, 2\};$$

a possible blocking set will be $\{(i, 1) \mid 1 \leq i \leq m\}$. We describe the cycles of the decomposition in the case $k = 0$ first. These are:

$$B = \{(\infty, (i,1), (i,2)) \mid 1 \leq i \leq m\},$$
$$\{((i,1), (j,2), (i,2), (j,1)) \mid 1 \leq i < j \leq m\}.$$

Now the case $k > 0$ is constructed from this. For clarity we describe the case $k = 1$ first. Remove from B the cycles

$$\{(\infty, (i,1), (i,2)) \mid i = 1, 2, 3, 4\} \cup \{((i,1), (j,2), (i,2), (j,1)) \mid 1 \leq i < j \leq 4\}$$

and replace them with a 4-cycle system of order 9, on the set

$$\{\infty\} \cup \{(i, j) \mid 1 \leq i \leq 4,\ j = 1, 2\}.$$

The result is a $\{C_3, C_4\}$-design of order $n = 2m+1$ and type $(m-4, m(m-1)/2 + 3)$.

Now for arbitrary k with $4k \leq m$, we take the cycles in B and remove $4k$ cycles of length 3, say, $\{(\infty, (i,1), (i,2)) \mid 1 \leq i \leq 4k\}$, and $6k$ cycles of length 4, namely $\{((i,1), (j,2), (i,2), (j,1)) \mid i < j\}$, for $i, j \in \{4s-3, 4s-2, 4s-1, 4s\}$, for each $s = 1, 2, \ldots, k$. Then we replace these with the cycles from k 4-cycle systems of order 9 on the vertex sets

$$\{\infty\} \cup \{(i, j) \mid i = 4s-3, 4s-2, 4s-1, 4s,\ j = 1, 2\}$$

for each $s = 1, 2, \ldots, k$, thus removing $4k$ cycles of length 3 and adding $3k$ to the number of cycles of length 4.

The result is a 2-coloured $\{C_3, C_4\}$-design of type $(p, q) = (m - 4k, m(m-1)/2 + 3k)$ for each $4k \leq m$.

This completes the proof of the theorem. □

References

[1] K. Heinrich, P. Horák and A. Rosa, *On Alspach's conjecture*, Discrete Math. **77** (1989), 97–121.

[2] S. Milici and Z. Tuza, *Cycle systems without 2-colorings*, J. Combinatorial Designs 4 (1996), 135–142.