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**The Mutual Intersections of Three Distinct 1-Factorizations**

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**Abstract.** In this paper, we show that for every $k \in I_3[v] = \{0, 1, 2, \ldots, \frac{v(v-1)}{2} - 6\}$, there exist three distinct 1-factorizations of $K_v$ which have exactly $k$ edges in common for each $v \geq 8$ and $v$ is even.

## 1. Introduction

A 1-factor of $K_{2m}$ is a collection of 2-element subsets (edges) of the vertex set $V(K_{2m})$ such that each vertex in $V(K_{2m})$ is contained in precisely one of these edges. A 1-factorization of $K_{2m}$ is a partition of the edges of $K_{2m}$ into 1-factors.

Let $S$ be a set of size $v$. A latin square of order $v$ based on $S$ is a $v \times v$ matrix with the property that each $i \in S$ occurs in each row and each column exactly once. A latin square $L = [\ell_{i,j}]$ is commutative (or symmetric) provided $\ell_{i,j} = \ell_{j,i}$ for all $i, j \in S$. If $\ell_{i,i} = c$ for all $i \in S$ and $c \in S$, then $L$ is called a constant diagonal latin square or a unipotent latin square. For brevity, we shall refer to a commutative unipotent latin square of order $v$ as CULS$(v)$. We say that three latin squares of order $v$, $L = [\ell_{i,j}]$, $M = [m_{i,j}]$ and $N = [n_{i,j}]$ have $k$ entries in common provided that there are exactly $k$ cells $(i, j)$ for which $\ell_{i,j} = m_{i,j} = n_{i,j}$. We avoid redundant counting by saying that three CULS$(v)$, $L, M$ and $N$ have $k$ entries in common, denoted by $|L \cap M \cap N| = k$ if there are exactly $k$ entries in common on the upper triangular parts (exclude the diagonal) of $L, M$ and $N$. Unless we mention otherwise, the CULS$(v)$ is based on the set $\{1, 2, \ldots, v-1\}$, $v$ is even and 0 in the diagonal.

It is well-known that a CULS$(v)$ is equivalent to a 1-factorization $F$ of $K_v$. (For a nice survey of results on 1-factorizations of $K_v$, the reader is refered to the article by Mendelsohn and Rosa[4]). Hence, instead of finding the mutual intersections of three distinct 1-factorzations, we do it for three distinct commutative unipotent latin squares.

In [1,3], it has been shown that for each $k \in \{0, 1, 2, \ldots, \frac{v(v-1)}{2} - 6, \frac{v(v-1)}{2} - 4, \frac{v(v-1)}{2}\}$ ($v$ even, $v \geq 8$), there exist two 1-factorizations of $K_v$ which have exactly $k$ edges in common. If we allow two of the three 1-factorizations to be the same, then there is not much to do in the mutual intersections of three 1-factorizations. In this paper, we consider the mutual intersections of three distinct 1-factorizations (CULS($v$)), and we obtain the best result for $v \geq 8$. For completeness, we also check the cases $v = 4$ and 6, and we have answers for them.

## 2. The Main Theorems

Denote $J_3[v]$ as the set $\{k$: there exist three distinct CULS($v$) which have $k$ entries in common$\}$. Also, we let $I_3[v] = \{0, 1, 2, \ldots, t_v - 6\}$ where $t_v = v(v-1)/2$. It is not difficult to see that there don't exist three distinct CULS($v$) which have $h$ entries in common for each $h \in \{t_v - 5, t_v - 4, t_v - 3, t_v - 2, t_v - 1, t_v\}$, hence we have the following theorem. For completeness, we give a proof.

**Theorem 2.1.** $J_3[v] \subseteq I_3[v]$ *for every* $v$.

Proof: By [5], it was shown that there don't exist two distinct CULS($v$) which have $k$ entries in common for each $k \in \{t_v - 5, t_v - 3, t_v - 2, t_v - 1, t_v\}$ where $t_v = v(v-1)/2$. Hence, each element of $\{t_v - 5, t_v - 3, t_v - 2, t_v - 1, t_v\}$ does not belong to $J_3[v]$. Let $L_1$ and $L_2$ be two CULS($v$) which have $t_v - 4$ entries in common. Without loss of generality, suppose that $L_1$ contains a latin subsquare $\begin{array}{|c|c|}\hline a & b \\ \hline b & a \\ \hline\end{array}$ and $L_2$ contains $\begin{array}{|c|c|}\hline b & a \\ \hline a & b \\ \hline\end{array}$ corresponding to the same entries as $\begin{array}{|c|c|}\hline a & b \\ \hline b & a \\ \hline\end{array}$ in $L_1$ such that $|L_1 \cap L_2| = t_v - 4$. Since there exist exactly two latin squares of order 2 based on $\{a, b\}$, it is impossible to construct another distinct CULS($v$) $L_3$ such that $|L_1 \cap L_2 \cap L_3| = t_v - 4$. Therefore $J_3[v] \subseteq I_3[v]$.

Before we go any further we need the following notation. $A + B = \{a + b : a \in A \text{ and } b \in B\}$.

**Lemma 2.2.** *A CULS($v$) can be embedded in a CULS($2v$).[1]*

**Lemma 2.3.** *A CULS($v$) can be embedded in a CULS($2v + 2$).[1]*

With the above two lemmas, we are ready to prove our main recursive theorems.

**Theorems 2.4.** *If* $J_3[v] = I_3[v]$ *and* $v \geq 8$, *then* $J_3[2v] = I_3[2v]$.

Proof: Since a CULS($v$) can be embedded in a CULS($2v$), we can of course embed three distinct CULS($v$), $A, B$ and $C$, in three distinct CULS($2v$), $L$, $M$ and $N$ (Figure 2.1). As a result of the embedding, $A'.B'$ and $C'$ are CULS($v$), $A_1, B_1$ and $C_1$ are latin squares of order $v$ based on $\{v, v+1, \ldots, 2v-1\}$. Moreover, we can permute the entries of a latin square ($A_1$ or $B_1$ or $C_1$) without affecting the embedding. Hence we have $J_3[2v] \supseteq J_3[v]$(from $A, B, C$)+$J_3[v] \cup \{t_v\} + \{0, v, 2v, \ldots, (v-2)v, v^2\}$ (from $B_1, C_1$). By counting, this implies $J_3[2v] \supseteq I_3[2v]$. We conclude the proof by Theorem 2.1. (We note here, since $A, B$ and $C$ are distinct, we can let $A_1 = B_1 = C_1$ or $A' = B' = C'$.)

$$L: \begin{array}{|c|c|}\hline A & A_1 \\ \hline A_1^T & A' \\ \hline \end{array} \quad M: \begin{array}{|c|c|}\hline B & B_1 \\ \hline B_1^T & B' \\ \hline \end{array} \quad N: \begin{array}{|c|c|}\hline C & C_1 \\ \hline C_1^T & C' \\ \hline \end{array}$$

Figure 2.1

**Theorem 2.5.** *If* $J_3[v] = I_3[v]$ *and* $v \geq 8$, *then* $J_3[2v+2] = I_3[2v+2]$.

Proof: By Lemma 2.3, we have three CULS$(2v+2)$, $L, M$ and $N$, (Figure 2.2) where $A, B$ and $C$ are three distinct CULS$(v)$. Since the sets of entries $\{1,2,\ldots,v-1\}$ and $\{v, v+1,\ldots,2v+1\}$ (outside $B$ and $C$) can be permuted independently, hence we have $J_3[2v+2] \supseteq J_3[v] + \{0, \frac{1}{2}(v+2), (v+2), \frac{3}{2}(v+2),\ldots, \frac{1}{2}(v-3)(v+2), \frac{1}{2}(v-1)(v+2)\} + \{0, \frac{1}{2}(2v+2), (2v+2),\ldots, \frac{1}{2}v(2v+2), \frac{1}{2}(v+2)(2v+2)\}$. Similar to Theorem 2.4, we have $J_3[2v+2] \supseteq I_3[2v+2]$ by direct counting and we conclude the proof by Theorem 2.1

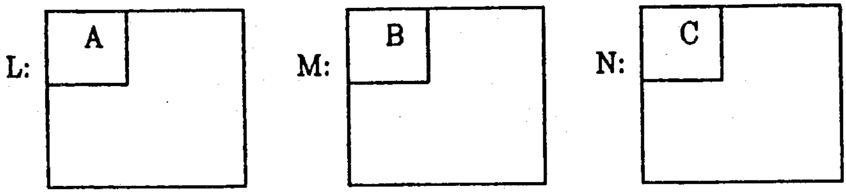


Figure 2.2

**Lemma 2.6.** $J_3[4] = I_3[4] = \{0\}$, $J_3[6] = \{0,1,2,3,4,5,6\}$.

Proof: It is easy to see $J_3[4] = \{0\}$, and $J_3[6]$ is obtained by checking all the 720 CULS(6), we put it in Appendix A.

**Lemma 2.7.** $J_3[8] = \{0,1,2,\ldots,22\} = I_8$.

Proof: Appendix B.

**Lemma 2.8.** $J_3[10] = I_3[10]$.

Proof: Appendix C.

**Lemma 2.9.** $J_3[12] = I_3[12]$.

Proof: We consider three distinct CULS(12) $L$, $M$, and $N$, as in Figure 2.1, where $A, B, C$ are three distinct CULS(6), $A', B'$, and $C'$ are CULS(6) and $A_1, B_1$, and $C_1$ are latin squares of order 6. In [2], it was shown that for each $k \in \{0,1,2,\ldots,30,32,36\}$, there exist two latin squares of order 6 which have exactly $k$ entries in common. Hence, by selecting suitable $B_1$ and $C_1$, we have $J_3[12] \supseteq \{0,1,2,\ldots,6\} + \{0,1,2,\ldots,30,32,36\} + \{0,1,\ldots,6,15\} = \{0,1,2,\ldots,57\}$. (We can let $A_1 = B_1$.) By Appendix D, $\{58,59,60\} \subseteq J_3[12]$, this implies $J_3[12] \supseteq I_3[12]$. By Theorem 2.1, we conclude the proof.

**Lemma 2.10.** $J_3[14] = I_3[14]$.

Proof: Since we can embed a CULS(6) in a CULS(14), we consider three distinct CULS(14) $L, M$ and $N$, as in Figure 2.2, and $A, B, C$ are CULS(6). Similar to the proof of Theorem 2.5, $J_3[14] \supseteq \{0,1,2,\ldots,6\} + \{0,4,8,12,20\} + \{0,7,14,21,\ldots,42,56\} = \{0,1,2,\ldots,74,76,\ldots,82\}$. By Appendix E, $\{75,83,84,85\} \subseteq J_3[14]$, this implies $J_3[14] \supseteq I_3[14]$. We conclude the proof by Theorem 2.1.

**Theorem 2.11.** $J_3[v] = I_3[v]$ *for every* $v \geq 8$.

Proof: By Lemma 2.6 to Lemma 2.10 and Theorem 2.4, Theorem 2.5.

**Remark**

The idea of constructing three 1-factorizations with prescribed intersections was mentioned to the author by Professor G. Quattrocchi in order to solve the intersection problem for three Steiner triple systems with prescribed 3-times repeated blocks and twice-repeated blocks. We deeply wish the problem can be solved right away.

## References


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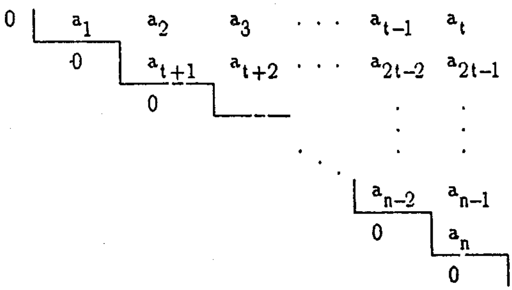
2. H.L. Fu, "On the construction of certain types of latin squares with prescribed intersections", Ph. D. Thesis, Auburn University, 1980.
3. C.C. Lindner and W.D. Wallis, *A note on 1-factorizations having a prescribed number of edges in common*, Annals of Discrete Mathematics **12** (1982), 203–209.
4. E. Mendelsohn and A. Rosa, *One-factorizations of the complete graph—a survey*, Journal of Graph Theory **9** (1985), 43–65.
5. T.M. Webb, "On idempotent commutative latin squares with a prescribed number of common entries", Ph. D. Thesis, Auburn University, 1980.

# Appendix A

In the list below, the row $i, a_1, a_2, a_3, \ldots, a_n$ represents the ith CULS($t+1$) whose entries above the diagonal are as in the following figure.




| # | entries of CULS(6) in the upper trianglar portion | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 3 | 4 | 5 | 2 | 5, | 1 | 4 | 2 | 1 | 3 |
| 2 | 1 | 2 | 3 | 4 | 5 | 3 | 5 | 2 | 4 | 4 | 5 | 1 | 1 | 2 | 3 |
| 3 | 1 | 2 | 3 | 4 | 5 | 4 | 2 | 5 | 3 | 5 | 3 | 1 | 1 | 4 | 2 |
| 4 | 1 | 2 | 3 | 4 | 5 | 4 | 5 | 3 | 2 | 1 | 5 | 3 | 2 | 4 | 1 |
| 7 | 1 | 2 | 3 | 5 | 4 | 3 | 4 | 2 | 5 | 5 | 4 | 1 | 1 | 2 | 3 |
| 10 | 1 | 2 | 3 | 5 | 4 | 4 | 5 | 2 | 3 | 1 | 3 | 5 | 4 | 2 | 1 |
| 11 | 1 | 2 | 3 | 5 | 4 | 5 | 2 | 4 | 3 | 4 | 3 | 1 | 1 | 5 | 2 |
| 13 | 1 | 2 | 4 | 3 | 5 | 3 | 2 | 5 | 4 | 5 | 4 | 1 | 1 | 3 | 2 |
| 22 | 1 | 2 | 4 | 5 | 3 | 4 | 5 | 3 | 2 | 3 | 1 | 5 | 2 | 1 | 4 |
| 37 | 1 | 3 | 2 | 4 | 5 | 2 | 4 | 5 | 3 | 5 | 1 | 4 | 3 | 1 | 2 |
| 145 | 2 | 1 | 3 | 4 | 5 | 3 | 4 | 5 | 1 | 5 | 2 | 4 | 1 | 2 | 3 |

| # entries in common | # of the above CULS(6) |
|---|---|
| 0 | 1,22,145 |
| 1 | 1,10,37 |
| 2 | 1,10,13 |
| 3 | 1,4,7 |
| 4 | 1,3,7 |
| 5 | 1,2,3 |
| 6 | 3,11,13 |

## Appendix B

| # | entries of CULS(8) in the upper trianglar portion |
|---|---|
| 1 | 1 2 3 4 5 6 7 3 2 5 6 7 4 1 6 7 4 5 7 4 5 6 1 2 3 3 2 1 |
| 2 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 1 2 3 3 2 1 |
| 3 | 3 4 5 6 7 1 2 5 4 7 1 2 6 7 1 2 6 3 2 6 3 1 3 4 5 5 4 7 |
| 4 | 7 6 5 4 3 2 1 5 4 3 2 1 6 3 2 1 7 4 1 7 6 2 6 5 7 4 5 3 |
| 5 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 1 3 2 2 3 1 |
| 6 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 3 2 1 1 2 3 |
| 7 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 3 1 2 2 1 3 |
| 8 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 5 4 7 6 4 5 3 2 1 1 2 3 |
| 9 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 5 4 7 6 4 5 3 1 2 2 1 3 |
| 10 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 7 6 5 4 6 7 4 5 3 2 1 1 2 3 |
| 11 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 7 6 5 4 6 7 4 5 3 1 2 2 1 3 |
| 12 | 1 2 3 4 5 6 7 3 2 5 4 7 6 6 7 1 4 5 1 7 5 4 6 3 2 2 3 1 |
| 13 | 1 2 3 4 5 6 7 3 2 5 4 7 6 6 7 1 4 5 1 7 5 4 6 2 3 3 2 1 |
| 14 | 1 2 3 4 5 6 7 3 2 5 4 7 6 6 7 1 5 4 1 7 4 5 6 3 2 2 3 1 |
| 15 | 1 2 3 4 5 6 7 3 2 5 4 7 6 6 7 1 5 4 1 7 4 5 6 2 3 3 2 1 |
| 16 | 1 2 3 4 5 6 7 3 4 7 2 5 6 1 6 7 4 5 5 6 7 2 3 2 1 1 4 3 |
| 17 | 1 2 3 4 5 6 7 3 4 5 6 7 2 6 7 1 5 4 1 7 2 5 2 3 6 4 3 1 |
| 18 | 1 2 3 4 5 6 7 3 4 5 6 7 2 6 7 4 1 5 2 7 5 1 1 3 6 2 3 4 |
| 19 | 1 2 3 4 5 6 7 3 2 5 7 4 6 1 6 4 7 5 7 6 5 4 1 2 3 3 2 1 |
| 20 | 1 2 3 4 5 6 7 3 2 5 6 7 4 4 7 1 5 6 6 7 1 5 2 3 1 4 3 2 |
| 21 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 2 1 3 3 1 2 |
| 22 | 1 2 3 4 5 6 7 3 2 5 4 7 6 4 7 6 5 1 6 7 1 5 1 3 2 2 3 4 |
| 23 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 7 6 5 4 6 7 4 5 2 3 1 1 3 2 |
| 24 | 1 2 3 4 5 6 7 3 2 5 4 7 6 4 6 7 1 5 7 6 5 1 1 2 3 3 2 4 |
| 25 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 7 6 4 5 6 7 5 4 1 2 3 3 2 1 |
| 26 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 5 4 7 6 4 5 2 3 1 1 3 2 |
| 27 | 1 2 3 4 5 6 7 3 4 7 2 5 6 1 6 7 4 5 5 6 7 2 3 2 1 1 4 3 |
| 28 | 1 2 3 4 5 6 7 3 4 7 2 5 6 1 6 7 4 5 5 6 7 2 1 2 3 3 4 1 |
| 29 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 5 4 7 6 4 5 1 2 3 3 2 1 |
| 30 | 1 2 3 4 5 6 7 3 2 5 4 7 6 1 6 7 4 5 7 6 5 4 2 3 1 1 3 2 |

| # entries in common | # of the above CULS(8) |
|---|---|
| 0 | 1,2,3 |
| 1 | 1,2,4 |
| 2 | 2,4,5 |
| 3 | 4,6,7 |
| 4 | 4,8,9 |
| 5 | 4,10,11 |
| 6 | 4,12,13 |
| 7 | 4,14,15 |
| 8 | 2,16,17 |
| 9 | 2,16,18 |
| 10 | 2,19,20 |
| 11 | 1,21,22 |
| 12 | 1,2,23 |
| 13 | 2,23,24 |
| 14 | 2,25,26 |
| 15 | 5,27,28 |
| 16 | 2,27,28 |
| 17 | 2,24,26 |
| 18 | 2,29,30 |
| 19 | 2,24,30 |
| 20 | 2,5,29 |
| 21 | 2,21,24 |
| 22 | 2,5,21 |

# Appendix C

| # | entries of CULS(10) in the upper triangular portion |
|---|---|
| 1 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 4 5 6 9 1 8 2 3 2 3 6 1 5 3 4 1 2 7 |
| 2 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 4 5 6 9 1 8 3 2 3 2 6 1 5 2 4 1 3 7 |
| 3 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 4 5 6 9 2 8 1 3 1 3 6 2 5 3 4 2 1 7 |
| 4 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 4 5 6 9 2 8 3 1 3 1 6 2 5 1 4 2 3 7 |
| 5 | 2 1 4 3 6 5 8 9 7 4 3 6 1 8 9 7 5 6 5 8 9 7 2 3 8 9 7 5 1 2 7 2 1 4 9 3 2 5 4 4 6 1 3 6 8 |
| 6 | 1 2 4 3 6 5 8 9 7 4 3 6 2 8 9 7 5 6 5 8 9 7 1 3 8 9 7 5 2 1 7 1 2 4 9 3 1 5 4 4 6 2 3 6 8 |
| 7 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 5 6 4 9 1 8 2 3 3 2 1 6 4 5 2 3 1 7 |
| 8 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 5 6 4 9 1 8 3 2 2 3 1 6 4 5 3 2 1 7 |
| 9 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 5 6 4 9 2 8 1 3 3 1 2 6 4 5 1 3 2 7 |
| 10 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 8 9 5 6 4 9 2 8 3 1 1 3 2 6 4 5 3 1 2 7 |
| 11 | 1 2 3 4 5 6 8 9 7 3 4 5 2 8 9 7 6 5 6 8 9 7 1 4 8 9 7 6 2 1 7 1 2 3 9 4 1 6 3 3 5 2 4 5 8 |
| 12 | 1 2 3 7 5 6 4 8 9 3 7 5 2 4 8 9 6 5 6 4 8 9 1 7 4 8 9 6 2 1 9 1 2 3 8 7 1 6 3 3 5 2 7 5 4 |
| 13 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 9 4 8 5 6 8 9 1 2 3 3 2 6 1 5 1 2 3 4 7 |
| 14 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 9 4 8 5 6 8 9 1 3 2 2 3 6 1 5 1 3 2 4 7 |
| 15 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 6 9 5 7 4 9 2 1 3 7 3 8 1 2 4 5 1 2 3 6 |
| 16 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 7 9 4 8 5 6 8 9 2 3 1 1 3 6 2 5 2 3 1 4 7 |
| 17 | 1 2 3 4 5 6 7 8 9 3 4 5 2 7 8 9 6 5 6 7 8 9 1 4 7 8 9.6 2 1 9 1 2 3 8 4 1 6 3 3 5 2 4 5 7 |
| 18 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 5 4 8 6 9 5 4 7 9 2 1 7 3 3 8 2 1 4 1 5 3 2 6 |
| 19 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 6 9 5 7 4 9 1 2 3 7 3 8 2 1 4 5 2 1 3 6 |
| 20 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 5 4 8 6 9 5 4 7 9 1 2 7 3 3 8 1 2 4 2 5 3 1 6 |
| 21 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 9 8 4.5 8 9 4 5 7 6 1 2 9 3 7 8·2 6 3 3 5 1 1 4 2 |
| 22 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 9 8 4 5 8 9 4 5 6 7 1 2 9 7 3 8 2 3 6 3 5 1 1 4 2 |
| 23 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 7 6 1 9 2 3 7 2 8 6 3 3 5 1 1 4 2 |
| 24 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 5 4 7 8 9 5 4 6 9.1 8 3 2 3 2 6 1 4 2 5 1 3 7 |
| 25 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 1 9 2 7 3 2 8 3 6 3 5 1 1 4 2 |
| 26 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 2 9 3 7 1 3 8 1 6 1 5 2 2 4 3 |
| 27 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 2 9 1 7 3 1 8 3 6 3 5 2 2 4 1 |
| 28 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 1 9 2 7 3 3 8 2 6 1 5 2 3 4 1 |
| 29 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 1 9 3 7 2 2 8 3 6 1 5 3 2 4 1 |
| 30 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 7 8 9 4 5 8 9 4 5 6 7 1 9 3 7 2 3 8 2 6 2 5 1 1 4 3 |
| 31 | 1 2 3 4 5 6 7 8 9 3 2 5 4 7 6 9 8 1 6 8 4 9 7 5 7 9 8 4 5 6 1 9 8 2 3 2 3 6 7 5 3 1 1 2 4 |
| 32 | 1 2 3 5 4 6 7 8 9 3 2 4 5 7 6 9 8 1 6 8 4 9 7 5 7 9 8 4 5 6 1 9 8 2 3 2 3 6 7 5 3 1 1 2 4 |
| 33 | 1 2 3 5 4 6 7 8 9 3 2 4 8 7 6 9 5 1 6 5 4 9 7 8 7 9 8 4 5 6 1 9 8 2 3 2 3 6 7 5 3 1 1 2 4 |
| 34 | 1 3 2 4 5 6 7 8 9 2 3 5 4 7 6 9 8 1 6 8 4 9 7 5 7 9 8 4 5 6 1 9 8 2 3 2 3 6 7 5 3 1 1 2 4 |
| 35 | 2 1 3 4 5 6 7 8 9 3 1 5 4 7 6 9 8 2 6 8 4 9 7 5 7 9 8 4 5 6 1 9 8 2 3 2 3 6 7 5 3 1 1 2 4 |

| #entries in common | #of the above CULS(10) |
|---|---|
| 0 | 1,2,5 |
| 1 | 2,4,5 |
| 2 | 2,5,7 |
| 3 | 5,9,10 |
| 4 | 5,7,9 |
| 5 | 5,9,15 |
| 6 | 5,15,18 |
| 7 | 5,21,22 |
| 8 | 6,19,20 |
| 9 | 3,7,11 |
| 10 | 1,3,11 |
| 11 | 1,2,11 |
| 12 | 11,14,16 |
| 13 | 8,12,13 |
| 14 | 1,12,13 |
| 15 | 2,12,14 |
| 16 | 3,7,12 |
| 17 | 1,3,12 |
| 18 | 1,2,12 |
| 19 | 8,13,17 |
| 20 | 1,13,17 |
| 21 | 2,14,17 |
| 22 | 3,7,17 |
| 23 | 1,3,17 |
| 24 | 1,2,17 |
| 25 | 2,17,24 |
| 26 | 1,8,13 |
| 27 | 1,9,13 |
| 28 | 1,7,13 |
| 29 | 7,9,13 |
| 30 | 1,4,7 |
| 31 | 1,2,7 |
| 32 | 23,25,26 |
| 33 | 23,25,27 |
| 34 | 23,25,28 |
| 35 | 23,25,30 |
| 36 | 1,2,3 |
| 37 | 25,28,29 |
| 38 | 31,32,33 |
| 39 | 31,34,35 |

# Appendix D

N(1)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 3 | 2 | 1 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 1 | 9 | 3 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 1 | 2 |
| 7 | 6 | 5 | 9 | 11 | 8 | 10 | 0 | 4 | 3 | 2 | 1 |
| 8 | 9 | 10 | 11 | 1 | 2 | 3 | 4 | 0 | 6 | 5 | 7 |
| 9 | 8 | 11 | 10 | 2 | 1 | 5 | 3 | 6 | 0 | 7 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 1 | 2 | 5 | 7 | 0 | 6 |
| 11 | 10 | 9 | 5 | 8 | 3 | 2 | 1 | 7 | 4 | 6 | 0 |

N(2)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 3 | 2 | 1 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 1 | 9 | 3 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 1 | 2 |
| 7 | 6 | 5 | 9 | 11 | 8 | 10 | 0 | 4 | 3 | 2 | 1 |
| 8 | 9 | 10 | 11 | 1 | 2 | 3 | 4 | 0 | 7 | 5 | 6 |
| 9 | 8 | 11 | 10 | 2 | 1 | 5 | 3 | 7 | 0 | 6 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 1 | 2 | 5 | 6 | 0 | 7 |
| 11 | 10 | 9 | 5 | 8 | 3 | 2 | 1 | 6 | 4 | 7 | 0 |

N(3)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 3 | 2 | 1 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 1 | 9 | 3 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 2 | 1 |
| 7 | 6 | 5 | 9 | 11 | 8 | 10 | 0 | 4 | 3 | 1 | 2 |
| 8 | 9 | 10 | 11 | 1 | 2 | 5 | 3 | 0 | 6 | 5 | 7 |
| 9 | 8 | 11 | 10 | 2 | 1 | 5 | 3 | 6 | 0 | 7 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 2 | 1 | 5 | 7 | 0 | 6 |
| 11 | 10 | 9 | 5 | 8 | 3 | 1 | 2 | 7 | 4 | 6 | 0 |

N(4)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 3 | 2 | 1 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 3 | 9 | 1 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 1 | 2 |
| 7 | 6 | 5 | 9 | 11 | 8 | 10 | 0 | 4 | 1 | 2 | 3 |
| 8 | 9 | 10 | 11 | 1 | 2 | 3 | 4 | 0 | 6 | 5 | 7 |
| 9 | 8 | 11 | 10 | 2 | 3 | 5 | 1 | 6 | 0 | 7 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 1 | 2 | 5 | 7 | 0 | 6 |
| 11 | 10 | 9 | 5 | 8 | 1 | 2 | 3 | 7 | 4 | 6 | 0 |

N(5)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 2 | 3 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 3 | 2 | 0 | 1 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 2 | 3 | 1 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 1 | 9 | 3 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 1 | 2 |
| 7 | 6 | 5 | 9 | 11 | 3 | 10 | 0 | 4 | 3 | 2 | 1 |
| 8 | 9 | 10 | 11 | 1 | 2 | 3 | 4 | 0 | 6 | 5 | 7 |
| 9 | 8 | 11 | 10 | 2 | 1 | 5 | 3 | 6 | 0 | 7 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 1 | 2 | 5 | 7 | 0 | 6 |
| 11 | 10 | 9 | 5 | 8 | 3 | 2 | 1 | 7 | 4 | 6 | 0 |

N(6)=

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 2 | 1 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 0 | 1 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 |
| 2 | 1 | 0 | 3 | 6 | 7 | 4 | 5 | 10 | 11 | 8 | 9 |
| 1 | 2 | 3 | 0 | 7 | 6 | 8 | 9 | 11 | 10 | 4 | 5 |
| 4 | 5 | 6 | 7 | 0 | 10 | 9 | 11 | 1 | 2 | 3 | 8 |
| 5 | 4 | 7 | 6 | 10 | 0 | 11 | 8 | 2 | 1 | 9 | 3 |
| 6 | 7 | 4 | 8 | 9 | 11 | 0 | 10 | 3 | 5 | 1 | 2 |
| 7 | 6 | 5 | 9 | 11 | 8 | 10 | 0 | 4 | 3 | 2 | 1 |
| 8 | 9 | 10 | 11 | 1 | 2 | 3 | 4 | 0 | 6 | 5 | 7 |
| 9 | 8 | 11 | 10 | 2 | 1 | 5 | 3 | 6 | 0 | 7 | 4 |
| 10 | 11 | 8 | 4 | 3 | 9 | 1 | 2 | 5 | 7 | 0 | 6 |
| 11 | 10 | 9 | 5 | 8 | 3 | 2 | 1 | 7 | 4 | 6 | 0 |

N(1), N(2), and N(3) have 58 entries in common
N(1), N(3), and N(4) have 59 entries in common
N(1), N(5), and N(6) have 60 entries in common

# Appendix E

N(1) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 4 | 5 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 5 | 4 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(2) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 2 | 3 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 3 | 2 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 2 | 3 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 4 | 5 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 5 | 4 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(3) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 2 | 1 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 3 | 0 | 1 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 1 | 0 | 3 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 1 | 2 | 3 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 4 | 5 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 5 | 4 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(4) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 4 | 5 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 2 | 9 | 3 |
| 5 | 4 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 3 | 8 | 2 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 2 | 3 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 3 | 2 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(5) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 4 | 5 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 5 | 4 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 3 | 8 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 8 | 3 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 3 | 1 | 8 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 8 | 9 | 3 | 10 | 4 | 6 | 1 | 5 | 0 |

N(6) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 4 | 7 | 6 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 3 | 2 | 4 | 5 | 6 | 7 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 5 | 4 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 4 | 5 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 7 | 6 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 6 | 7 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(7) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 4 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 0 | 3 | 2 | 4 | 5 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 5 | 4 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 4 | 5 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 8 | 9 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 9 | 8 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 10 | 11 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 11 | 10 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(8) =

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 4 | 6 | 7 | 9 | 8 | 11 | 10 | 12 | 13 |
| 1 | 0 | 3 | 2 | 4 | 5 | 7 | 6 | 8 | 9 | 10 | 11 | 13 | 12 |
| 2 | 3 | 0 | 1 | 6 | 9 | 5 | 11 | 4 | 12 | 8 | 13 | 10 | 7 |
| 3 | 2 | 1 | 0 | 8 | 7 | 10 | 4 | 13 | 5 | 12 | 9 | 6 | 11 |
| 5 | 4 | 6 | 8 | 0 | 12 | 13 | 10 | 7 | 11 | 1 | 3 | 9 | 2 |
| 4 | 5 | 9 | 7 | 12 | 0 | 11 | 1 | 6 | 10 | 13 | 2 | 8 | 3 |
| 6 | 7 | 5 | 10 | 13 | 11 | 0 | 12 | 3 | 2 | 4 | 8 | 1 | 9 |
| 7 | 6 | 11 | 4 | 10 | 1 | 12 | 0 | 2 | 13 | 9 | 5 | 3 | 8 |
| 9 | 8 | 4 | 13 | 7 | 6 | 3 | 2 | 0 | 1 | 5 | 12 | 11 | 10 |
| 8 | 9 | 12 | 5 | 11 | 10 | 2 | 13 | 1 | 0 | 3 | 6 | 7 | 4 |
| 11 | 10 | 8 | 12 | 1 | 13 | 4 | 9 | 5 | 3 | 0 | 7 | 2 | 6 |
| 10 | 11 | 13 | 9 | 3 | 2 | 8 | 5 | 12 | 6 | 7 | 0 | 4 | 1 |
| 12 | 13 | 10 | 6 | 9 | 8 | 1 | 3 | 11 | 7 | 2 | 4 | 0 | 5 |
| 13 | 12 | 7 | 11 | 2 | 3 | 9 | 8 | 10 | 4 | 6 | 1 | 5 | 0 |

N(1), N(6), and N(8) have 75 entries in common
N(1), N(6), and N(7) have 83 entries in common
N(1), N(4), and N(5) have 84 entries in common
N(1), N(2), and N(3) have 85 entries in common