20. On a Certain Result of Z. Opial

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1. **Introduction.** In a recent paper [1], Z. Opial proved the following interesting integral inequality:

Theorem. *Let* $y(x)$ *be of class* C' *on* $0\leqslant x\leqslant h$, *and satisfy* $y(0)=y(h)=0$, $y(x)>0$ *on* $(0, h)$. *Then*

$$\int_0^h |yy'|\,dx\leqslant\frac{h}{4}\int_0^h y'^2dx. \tag{1}$$

The constant $h/4$ *is best possible.*

C. Olech [2] showed that (1) is valid for any function which is absolutely continuous on $[0, h]$, and satisfies the boundary conditions $y(0)=y(h)=0$, and Olech's proof of (1) was much simpler than that of Opial. P. R. Beesack [3] gave an even simpler proof of (1) under the hypotheses of Olech, and he also gave more general inequalities of the same type. Later, many simpler proofs were given by N. Levinson [4], C. L. Mallows [5], and R. N. Pederson [6].

By Mallows' method of the proof of (1) we shall give a simple proof of some results of Beesack [3], and show how this method can be used to yield generalization of Opial's and Beesack's inequalities.

2. On the inequality $2\int_a^b |yy'|\,dx\leqslant K\int_a^b py'^2dx$.

Let us define $z(x)=\int_a^x |y'(t)|\,dt$, $a\leqslant x\leqslant X$. Then $|y(x)|\leqslant z(x)$ for $a\leqslant x\leqslant X$, and we have

$$2\int_a^X |y(x)y'(x)|\,dx\leqslant 2\int_a^X zz'dx=z^2(X).$$

Now by the definition of $z(x)$ and Schwarz's inequality

$$z^2(X)=\left(\int_a^X |y'(x)|\,dx\right)^2\leqslant\int_a^X p^{-1}(x)dx\int_a^X py'^2dx.$$

There is equality only if $y=A\int_a^x p^{-1}(t)dt$, A being a constant. Similarly, define $z(x)=-\int_x^b |y'(t)|\,dt$, $X\leqslant x\leqslant b$. Then $|y(x)|\leqslant -z(x)$ for $X\leqslant x\leqslant b$, and

$$2\int_X^b |yy'|\,dx\leqslant 2\int_X^b -zz'dx=z^2(X)=\left(-\int_X^b |y'|\,dx\right)^2\leqslant\int_X^b p^{-1}dx\int_X^b py'^2dx.$$

There is equality only if $y=B\int_x^b p^{-1}(t)dt$, with B constant. Now, we take X such that

$$K=\int_a^X p^{-1}(x)dx=\int_X^b p^{-1}(x)dx, \tag{2}$$

then we get

Theorem 1. *Let $p(x)$ be a positive and continuous function on a finite or infinite interval $a<x<b$, such that $\int_a^b p^{-1}(x)dx<\infty$ and let $y(x)$ be an absolutely continuous function on (a, b) with $y(a)=y(b)=0$. Then*

$$2\int_a^b |yy'|\,dx \leqslant K\int_a^b py'^2dx,$$

where K is defined by (2). *Equality holds only if*

$$y(x)=A\int_a^x p^{-1}(t)dt \quad (a\leqslant x\leqslant X), \qquad y(x)=B\int_x^b p^{-1}(t)dt \quad (X\leqslant x\leqslant b).$$

Opial's inequality (1) is a special case of Theorem 1 that $a=0$, $b=h$, and $p(x)=1$.

3. On the inequality $2\int_a^b q\,|yy'|\,dx\leqslant\int_a^b p^{-1}dx\int_a^b pqy'^2dx$.

Lemma 1. *Let $p(x)$ be a bounded, positive and non-increasing function defined on $a\leqslant x\leqslant b$. Let $y(x)$ be absolutely continuous on $a\leqslant x\leqslant b$, with $y(a)=0$. Then*

$$\int_a^b p\,|yy'|\,dx\leqslant\frac{b-a}{2}\int_a^b py'^2dx. \tag{3}$$

Proof. Define $z(x)=\int_a^x\frac{1}{\sqrt{p(t)}}|y'(t)|\,dt$ $(a\leqslant x\leqslant b)$. Then

$$|y(x)|=\left|\int_a^x y'(t)dt\right|\leqslant\frac{1}{\sqrt{p(x)}}\int_a^x\sqrt{p(t)}\,|y'(t)|\,dt=\frac{z(x)}{\sqrt{p(x)}}$$

for $a\leqslant x\leqslant b$, so that

$$2\int_a^b p\,|yy'|\,dx\leqslant 2\int_a^b zz'dx=z^2(b)=\left(\int_a^b\sqrt{p(x)}\,|y'(x)|\,dx\right)^2.$$

By Schwarz's inequality,

$$\left(\int_a^b\sqrt{p(x)}\,|y'(x)|\,dx\right)^2\leqslant\int_a^b dx\int_a^b py'^2dx=(b-a)\int_a^b py'^2dx.$$

There is equality only it $p=$constant and $y=Ax$ with A constant.

Lemma 2. *Let $p(x)$ be a bounded, positive and non-decreasing function defined on $a\leqslant x\leqslant b$, and let $y(x)$ be an absolutely continuous function on $a\leqslant x\leqslant b$, with $y(b)=0$. Then the inequality* (3) *holds. Moreover, there is equality only if $p=$constant, $y=B(x-b)$, with B constant.*

Proof. Define $z(x)=-\int_x^b\sqrt{p(t)}\,|y'(t)|\,dt$ $(a\leqslant x\leqslant b)$. Then $|y(x)|\leqslant-\frac{z(x)}{\sqrt{p(x)}}$ for $a\leqslant x\leqslant b$, so that

$$2\int_a^b p\,|yy'|\,dx\leqslant-2\int_a^b zz'dx=z^2(a)=\left(\int_a^b\sqrt{p(x)}\,|y'(x)|\,dx\right)^2.$$

By Schwarz's inequality (3) follows immediately. There is equality only if $p=$constant, $y=B(x-b)$ with B constant.

From Lemma 1 and Lemma 2, follows immediately

Theorem 2. *Let $p(x)$ be a bounded, positive and monotonic function defined on $a \leqslant x \leqslant b$, and let $y(x)$ be an absolutely continuous function on $a \leqslant x \leqslant b$, with $y(a)=y(b)=0$. Then the inequality* (3) *holds. If $p=$constant, then the constant $1/2$ can be replaced by $1/4$, and then Opial's inequality* (1) *is obtained by letting $a=0$ and $b=h$.*

We shall now prove a generalization of Beesack's theorem.

Theorem 3. *Let $p(x)$ be positive on $a \leqslant x \leqslant X$, with $\int_a^X p^{-1}dx < \infty$, and let $q(x)$ be bounded, positive and non-increasing on $a \leqslant x \leqslant X$; $y(x)$ be any function which is absolutely continuous on $a \leqslant x \leqslant X$, with $y(a)=0$. Then*

$$(4) \qquad 2\int_a^X q\,|yy'|\,dx \leqslant \int_a^X p^{-1}dx \int_a^X pqy'^2 dx.$$

There is equality only if $q=$ constant, $y=A\int_a^x p^{-1}(t)dt$ or $y=0$.

Proof. Define $z(x)=\int_a^x \sqrt{q(t)}\,|y'(t)|\,dt$. Then $z'(x)=\sqrt{q(x)}\,|y'(x)|$ for $a \leqslant x \leqslant X$. Since $q(x)$ is non-increasing on $a \leqslant x \leqslant X$, we have

$$|y(x)| \leqslant \int_a^x |y'(t)|\,dt \leqslant \frac{1}{\sqrt{q(x)}}\int_a^x \sqrt{q(t)}\;|y'(t)|\,dt = z(x)q(x)^{-1/2}.$$

Hence

$$2\int_a^X q\,|yy'|\,dx \leqslant 2\int_a^X zz'dx = z^2(X) = \left(\int_a^X \sqrt{q(t)}\;|y'(t)|\,dt\right)^2.$$

By Schwarz's inequality, we get (4). There is equality only if $q=$ constant or $y=0$.

Similarly, we have

Theorem 3′. *Let $p(x)$ be positive on $X \leqslant x \leqslant b$, with $\int_X^b p^{-1}dx < \infty$, and let $q(x)$ be bounded, positive and non-decreasing on $X \leqslant x \leqslant b$; $y(x)$ be any function which is absolutely continuous on $X \leqslant x \leqslant b$, with $y(b)=0$. Then*

$$(5) \qquad 2\int_X^b q\,|yy'|\,dx \leqslant \int_X^b p^{-1}dx \int_X^b pqy'^2 dx.$$

Moreover, there is equality only if $q=$constant or $y=0$.

Theorem 1 is a special case of the combination of Theorem 3 and Theorem 3′, taking $q=$constant.

4. On the inequality $(m+n)\int_a^b |y^m y'^n|\,dx \leqslant n(b-a)^m \int_a^b |y'|^{m+n}dx$.

Lemma 3. *Let $y(x)$ be absolutely continuous on $a \leqslant x \leqslant X$, with $y(a)=0$. Then*

$$(6) \qquad (n+1)\int_a^X |y^n y'|\,dx \leqslant (X-a)^n \int_a^X |y'(x)|^{n+1}dx, \quad n \geq 1.$$

Moreover, equality holds only if $y=A(x-a)$, with A constant.

Proof. Define $z(x)=\int_a^x |y'(t)|\,dt$. Then $|y(x)|\leqslant z(x)$ for $a\leqslant x\leqslant X$, and we have

$$(n+1)\int_a^X |y^n y'|\,dx\leqslant(n+1)\int_a^X z^n z' dx=z^{n+1}(X)=\left(\int_a^X |y'(t)|\,dt\right)^{n+1}.$$

By Hölder's inequality, (6) follows immediately. There is equality only if $y=A(x-a)$, with A constant.

Lemma 4. *Let $y(x)$ be an absolutely continuous function on $a\leqslant x\leqslant X$, with $y(b)=0$, then*

$$(n+1)\int_X^b |y^n y'|\,dx\leqslant(b-X)^n\int_X^b |y'|^{n+1}dx,\quad n\geqslant 1. \tag{7}$$

Moreover, equality holds only if $y=B(x-b)$.

Proof. Define $z(x)=-\int_x^b |y'(t)|\,dt$. Then $z'(x)=|y'(x)|$ for $X\leqslant x\leqslant b$, and $|y(x)|\leqslant -z(x)$. Hence

$$(n+1)\int_X^b |y^n y'|\,dx\leqslant(n+1)\int_X^b (-z)^n z' dx=(-z(X))^{n+1}=\left(\int_X^b |y'|\,dt\right)^{n+1}.$$

By Hölder's inequality, (7) follows immediately.

Take $X=(a+b)/2$ in Lemma 3 and Lemma 4, then we have

Theorem 4. *Let $y(x)$ be an absolutely continuous function on $a\leqslant x\leqslant b$, with $y(a)=y(b)=0$. Then*

$$2^n(n+1)\int_a^b |y^n y'|\,dx\leqslant(b-a)^n\int_a^b |y'|^{n+1}dx,\quad n\geqslant 1. \tag{8}$$

We note that Opial's inequality (1) is the special case with $n=1$, $a=0$, and $b=h$.

Corollarly. *Let $y(x)$ be as in Theorem 4, and let $P(y)=\sum_{k=1}^{n} a_k y(x)^k$, with $a_k\geqslant 0$, $k=1, 2, \cdots n$. Then*

$$\int_a^b |P(y(x))'|\,dx\leqslant\frac{2}{b-a}\int_a^b P\left(\frac{b-a}{2}|y'|\right)dx. \tag{9}$$

Example. Let $y(x)=x(a-x)$, with $0<a<\sqrt{2}$, and let $P(y)=\sum_{k=1}^{\infty} y^k(x)$. Then the relation (9) becomes $\dfrac{(x-1)(2+3x)}{2x(x+1)}\leqslant\log x$ in the interval $(1, \infty)$.

Lemma 5. *Let $y(x)$ be an absolutely continuous function on $a\leqslant x\leqslant X$, with $y(a)=0$, then*

$$(n+1)\int_a^X |yy'^n|\,dx\leqslant n(X-a)\int_a^X |y'|^{n+1}dx,\quad n\geqslant 1. \tag{10}$$

Proof. Define $z(x)=\int_a^x |y'(t)|^n dt$. Then $z'(x)=|y'(x)|^n$ for $a\leqslant x\leqslant X$, and by Hölder's inequality

$$|y(x)|\leqslant\int_a^x |y'(t)|\,dt\leqslant\left(\int_a^x dt\right)^{(n-1)/n}\left(\int_a^x |y'|^n dt\right)^{1/n}\leqslant(X-a)^{(n-1)/n}(z(x))^{1/n}.$$

Hence

$$\begin{aligned}(n+1)\int_a^X |yy'^n|\,dx &\leqslant(n+1)\int_a^X (X-a)^{(n-1)/n} z^{1/n} z' dx\\ &=n(X-a)^{(n-1)/n}(z(X))^{(n+1)/n}.\end{aligned}$$

By Hölder's inequality, (10) follows immediately.

Lemma 6. *If $y(x)$ is absolutely continuous on $X \leqslant x \leqslant b$, with $y(b)=0$, then*

$$(n+1)\int_X^b |yy'^n|\,dx \leqslant n(b-X)\int_X^b |y'|^{n+1}dx, \quad n \geqslant 1. \tag{11}$$

Proof. Define $z(x)=-\int_x^b |y'(t)|^n dt$. Then $z'(x)=|y'(x)|^n$ for $X \leqslant x \leqslant b$, and then

$$|y(x)| \leqslant \int_x^b |y'(t)|\,dt \leqslant (b-X)^{(n-1)/n}(-z(x))^{1/n}.$$

Hence

$$(n+1)\int_X^b |yy'^n|\,dx \leqslant (n+1)\int_X^b (b-X)^{(n-1)/n}(-z)^{1/n}z'dx$$
$$=n(b-X)^{(n-1)/n}(-z(X))^{(n+1)/n}.$$

Now,

$$(-z(X))^{(n+1)/n}=\left(\int_X^b |y'|^n\,dx\right)^{(n+1)/n} \leqslant \left(\int_X^b dx\right)^{1/n}\int_X^b |y'|^{n+1}dx$$
$$=(b-X)^{1/n}\int_X^b |y'(x)|^{n+1}dx.$$

Therefore (11) follows immediately.

If we take $X=(a+b)/2$ in Lemma 5 and Lemma 6, then we have the following

Theorem 5. *If $y(x)$ is absolutely continuous on $a \leqslant x \leqslant b$, with $y(a)=y(b)=0$. Then*

$$\int_a^b |yy'^n|\,dx \leqslant \frac{n(b-a)}{2(n+1)}\int_a^b |y'(x)|^{n+1}dx, \quad n \geqslant 1. \tag{12}$$

We observe that Opial's inequality (1) is a special case obtained by taking $n=1$, $a=0$, and $b=h$.

In order to generalize Theorems 4 and 5 we prove the following lemmas.

Lemma 7. *If $y(x)$ is absolutely continuous on $a \leqslant x \leqslant X$, with $y(a)=0$. Then*

$$(m+n)\int_a^X |y^m y'^n|\,dx \leqslant n(X-a)^m\int_a^X |y'(x)|^{m+n}dx, \quad m, n \geqslant 1. \tag{13}$$

Proof. Define $z(x)=\int_a^x |y'(t)|^n dt$. Then $z'(x)=|y'(x)|^n$ for $a \leqslant x \leqslant X$, and then

$$|y(x)| \leqslant \int_a^x |y'(t)|\,dt \leqslant \left(\int_a^x dt\right)^{(n+1)/n}\left(\int_a^x |y'(t)|^n dt\right)^{1/n} \leqslant (X-a)^{(n-1)/n}(z(x))^{1/n}.$$

Hence

$$(m+n)\int_a^X |y^m y'^n|\,dx \leqslant (m+n)\int_a^X (X-a)^{m(n-1)/n}z^{m/n}z'dx$$
$$=n(X-a)^{m(n-1)/n}(z(X))^{(m+n)/n}.$$

Thus (13) follows immediately.

Lemma 8. *If $y(x)$ is absolutely continuous on $X \leqslant x \leqslant b$, with $y(b)=0$. Then*

$$(14) \qquad (m+n)\int_X^b |y^m y'^n|\,dx \leqslant n(b-X)^m \int_X^b |y'|^{m+n}dx, \quad m, n \geqslant 1.$$

Proof. Define $z(x) = -\int_x^b |y'(t)|^n dt$. Then $z'(x) = |y'(x)|^n$ for $X \leqslant x \leqslant b$, and

$$|y(x)| \leqslant \int_x^b |y'(t)|\,dt \leqslant (b-X)^{(n-1)/n}(-z(x))^{1/n}.$$

Hence

$$(m+n)\int_X^b |y^m y'^n|\,dx \leqslant (m+n)\int_X^b (b-X)^{m(n-1)/n}(-z)^{m/n}z'dx$$
$$= n(b-X)^{m(n-1)/n}(-z(X))^{(m+n)/n}.$$

Thus (14) follows immediately.

If we take $X=(a+b)/2$ in Lemma 7 and Lemma 8, we have

Theorem 6. *If $y(x)$ is absolutely continuous on $a \leqslant x \leqslant b$, with $y(a)=y(b)=0$, then*

$$\int_a^b |y^m y'^n|\,dx \leqslant \frac{n}{m+n}\left(\frac{b-a}{2}\right)^m \int_a^b |y'|^{m+n}dx, \quad m, n \geqslant 1.$$

Opial's inequality (1) is a special case that $m=n=1$, $a=0$, and $b=h$.

References

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