NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 31 (2002), 99–107

TWO–COLORABLE $\{C_4, C_k\}$–DESIGNS

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(Received October 2000)

Abstract. In this paper, we show that there exists a 2–colorable $\{C_4, C_k\}$–design of order n for each $k \geq 3$ and for each admissible order n of a $\{C_4, C_k\}$–design.

1. Introduction

Let K_n be the complete graph on the set of n vertices $V_n = \{1, 2, \cdots, n\}$ with the set of $\binom{n}{2}$ edges, E_n, which join all possible pairs of vertices in V_n. A G–design of order n is an edge–disjoint decomposition of K_n into copies of the graph $G = \big(V(G), E(G)\big)$. The number n is called an *admissible* order of such a G–design. For example, if $G = K_k$ then the G–design is a 2–$(n, k, 1)$ balanced incomplete block design (BIBD) in the usual notation, and in particular if $k = 3$ then the G–design is a Steiner triple system of order n. Again if $G = C_k$, a cycle of k edges, then the G–design is called a *cycle design* or *cycle system*. Since K_3 and C_3 are the same graph, a Steiner triple system of order n is also a cycle system.

Here we consider a slightly more general situation where cycles of lengths 4 and k are both allowed. In the process of constructing a $\{C_4, C_k\}$–design with the properties we want, we use a more general structure again, namely a $\{C_4, K_k\}$–design.

The isomorphic copies of the graphs that occur in the partition are called *blocks* of the design. In an unfortunate clash of well–established terminology, a proper subset S of V_n is said to be a *blocking set* of the G-design provided that the vertex set of each of its blocks contains at least one element of S, but is not contained in S. Thus if we color S and $V_n \setminus S$ with two distinct colors, the vertex set of every block contains at least one vertex of each color. For convenience, we call a design with a blocking set a 2–*colorable design*. Quite a number of 2–colorable designs have been obtained so far; for example, see [**2**].

Since we use the following result several times, we state it here; for completeness, we include a proof.

Theorem 1.1. *Let $n \equiv 1 \pmod 8$. Then there exists a 2–colorable 4–cycle system of order n with a blocking set of size $\frac{n-1}{2}$.*

Proof. Let $V(K_n) = \mathcal{Z}_n$. We prove the theorem by induction on n.

For $n = 9$, let $S = \{1, 3, 5, 7\}$ be the blocking set, let (a, b, c, d) denote the 4–cycle with edges ab, bc, cd, da, and let

$$T = \{(0 + i, 1 + i, 5 + i, 3 + i) \mid i \in \mathcal{Z}_9\}.$$

Then $(\mathcal{Z}_9, T)$ is the system we need.

2000 *AMS Mathematics Subject Classification:* 05B30, 05C70.

Now assume the assertion is true for $n = 8k+1$, $k \geq 1$. Let $S = \{1, 3, 5, \cdots, 8k-1\}$ be the blocking set of the 2–colorable 4–cycle system $(\mathcal{Z}_n, T_1)$. Next let $X = \{0, 8k+1, 8k+2, \cdots, 8k+8\}$, where (X, T_2) is a 2–colorable 4–cycle system with blocking set $\{8k+1, 8k+3, 8k+5, 8k+7\}$. Finally let

$$T_3 = \{(8k+i, j, 8k+i+1, j+1) \mid i = 1, 3, 5, 7; j = 1, 3, 5, \cdots, 8k-1\},$$

so that a 2–colorable 4–cycle system of order $n+8$, (X, T), can be obtained by taking $X = \mathcal{Z}_{8k+9}$ and $T = T_1 \cup T_2 \cup T_3$. Note here that $\{1, 3, 5, \cdots, 8k+7\}$ is a blocking set of size $4k+4$.

This concludes the proof. □

In this paper, we use the idea of *packing* to obtain our construction; see [**4**] for instance. A *packing* of K_n with 4–cycles is an ordered triple (V_n, P, L), where P is a collection of edge–disjoint 4–cycles of the edge–set E_n and $L \subseteq E_n$ is the set of edges not belonging to any 4–cycle in P. The number n is called the *order* of the packing and the set of edges L is called the *leave*.

First, in Section 2, we show that there exists a 2–colorable maximum packing of K_n with 4–cycles. Then in Section 3, for any odd integer $k \geq 3$ and for each admissible order n of a $\{C_4, K_k\}$–design, we construct a 2–colorable $\{C_4, K_k\}$–design of order n. Clearly, this implies the existence of a 2–colorable $\{C_4, C_k\}$–design with odd k. Finally, for each $k \geq 3$ and for each admissible order n of a $\{C_4, C_k\}$–design, we construct a 2–colorable $\{C_4, C_k\}$–design of order n.

2. 2–Colorable Maximum Packings with C_4

It is well–known (see for example [**4**]) that any maximum packing of K_n with copies of C_4 has leave a 1–factor for n even, and leave as shown in Table 1 for n odd. For convenience, such a packing will be denoted by MP4CS(n).

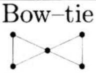
Order (mod 8)	1	3	5	7
Minimum Leave	∅	C_3	Bow–tie 	C_5

Table 1

Often we need the idea of a *balanced* 2–coloring. This is one in which the number of vertices colored 1 and the number of vertices colored 2 differ by at most one.

Theorem 2.1. *For each* $n \geq 1$, *there exists a* 2*–colorable* MP4CS(n). *If* $n = 2m$ *or if* $n = 2m+1$, *the blocking set has size* m.

Proof. If $n \equiv 1 \pmod 8$, then the proof follows by Theorem 1.1.

If n is even, let $n = 2m$, let $V(K_n) = \{a_i, b_i \mid i \in \mathcal{Z}_m\}$ and let $F = \{a_i b_i \mid i \in \mathcal{Z}_m\}$ be the leave. If we color a_i and b_i with 1 and 2 respectively for each $i \in \mathcal{Z}_m$ and let (a_i, a_j, b_i, b_j) be a 4–cycle for each unordered pair $\{i, j\}$, $i \neq j$ and $i, j \in \mathcal{Z}_m$, we obtain a 2–colorable MP4CS(n). This leaves three cases.

(i) $n \equiv 3 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{\infty_1, \infty_2\}$. By Theorem 1.1, we have a 2–colorable 4CS$(8k+1)$, $(\mathcal{Z}_{8k+1}, T)$, with 2–coloring ϕ such that

$$\phi(i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

The 4–cycles in T, together with $\{(\infty_1, 2j+1, \infty_2, 2j+2) \mid j = 0, 1, \cdots, 4k-1\}$, form a MP4CS$(n)$ with leave $C_3 = (0, \infty_1, \infty_2)$. The colors for ∞_1 and ∞_2 may be chosen arbitrarily, but we may as well color ∞_i with i, $i = 1, 2$, in order to obtain a balanced 2–colouring for a 2–colorable MP4CS(n).

(ii) $n \equiv 5 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{a, b, c, d\}$, and let the 2–colorable 4CS$(8k+1)$ be defined as in case (i). By a similar technique, we find that the 4–cycles in T, together with $\{(a, 2j+1, b, 2j+2), (c, 2j+1, d, 2j+2) | j = 0, 1, \cdots, 4k-1\}$ and (a, b, c, d), give an MP4CS(n) with leave $(0, a, c) \cup (0, b, d)$. Now coloring a, c with 2 and b, d with 1 gives a 2–colorable MP4CS(n), as required.

(iii) $n \equiv 7 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{a, b, c, d, e, f\}$. Again using a similar argument, we find that the 4–cycles in T, together with

$$\{(a, 2j+1, b, 2j+2), (c, 2j+1, d, 2j+2), (e, 2j+1, f, 2j+2) \mid j = 0, 1, \cdots, 4k-1\}$$

and $\{(0, a, d, c) \cup (a, c, f, e) \cup (0, b, a, f) \cup (b, e, 0, d)\}$, decompose $K_n \setminus C_5$ where the leave $C_5 = (b, c, e, d, f)$. Now the 2–colorable MP4CS(n) is obtained by coloring a, c, e with 1 and b, d, f with 2.

□

3. 2–Colorable $\{C_4, K_{2h+1}\}$–Designs

First, we need a lemma. Note that we use balanced colorings here.

Lemma 3.1. *If $n - m \equiv 1 \pmod 8$ and m is even, then there exists a 2–colorable $\{C_4, K_{m+1}\}$–design of order n.*

Proof. Let $(\mathcal{Z}_{n-m}, T)$ be a 2–colorable 4CS$(n-m)$ with 2–coloring ϕ such that

$$\phi(i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

Since m is even, let $m = 2s$ and let $\{c_1, d_1, c_2, d_2, \cdots, c_s, d_s\}$ be a set of m points. Now the cycles of T, together with the cycles in

$$\{(c_i, 2j+1, d_i, 2j+2) \mid j = 0, 1, \cdots, (n-m-3)/2, i = 1, 2, \cdots, s\}$$

and the complete graph based on $\{0, c_1, d_1, \cdots, c_s, d_s\}$, decompose K_n into 4–cycles and a K_{m+1}. If we color each c_i with color 1 and each d_i with color 2, then we have a 2–colorable $\{C_4, K_{m+1}\}$–design. □

Lemma 3.2. *Let n be an admissible order of a $\{C_4, K_{2h+1}\}$–design. Then $n \equiv 1$ or $5 \pmod 8$ if $h \equiv 2 \pmod 4$, $n \equiv 1 \pmod 8$ if $h \equiv 0 \pmod 4$, and n is odd if h is odd.*

Proof. Since C_4 is a 2–regular graph and K_{2h+1} is $2h$–regular, each vertex of K_n must have even degree and thus n is odd. If h is even, then the number of edges in K_{2h+1} is also even, implying that K_n must have an even number of edges and hence that $n \equiv 1$ or $5 \pmod 8$. Next, if $4 \mid h$, then the number of edges in K_{2h+1} is also a multiple of 4, and so is the number of edges in K_n. Thus $n \equiv 1 \pmod 8$. □

Lemma 3.3. *There exists a 2–colorable $\{C_4, K_{4l+1}\}$–design of order $n = 8k+5$ for all k and all odd l such that $n \geq 4l+1$.*

Proof. Since l is odd, $n-4l \equiv 1 \pmod 8$. By Lemma 3.1, a 2–colorable $\{C_4, K_{4l+1}\}$–design of order n exists. □

We note here that a 4CS(n) can be considered as a $\{C_4, K_{2h+1}\}$–design of order n with no blocks of size $2h+1$.

Next, we consider the 2–colorable $\{C_4, K_{4l+3}\}$–designs.

Lemma 3.4. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k+3$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. First, if l is even, then $n-(4l+2) \equiv 1 \pmod 8$. The proof follows by Lemma 3.1.

Next, if l is odd, then $4l+3 \equiv 7 \pmod 8$. Let $j = 4l+3$. Then $\binom{8k+3}{2} - i\binom{8j+7}{2} = \frac{1}{2}[(8k+3)(8k+2) - i(8j+7)(8j+6)]$ which is not a multiple of 4 for $i = 0, 1$ and 2. So if there exists a $\{C_4, K_{4l+3}\}$–design of order n, then the design must contain at least three blocks of size $4l+3$. This implies that $n \geq 3(4l+3)-2$. Let $l = 2l'+1$. By direct counting, $[n-2(4l+2)]-(4l+2) = n-3(4l+2) = (8k+3)-3(8l'+6) \equiv 1 \pmod 8$. Since $4l+2$ is even, there exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n-2(4l+2)$ by Lemma 3.1.

Now let the design of order $n-8l-4$ that we have just described be (X_1, T_1), where $X_1 = \{0, c_1, c_2, \cdots, c_{n-8l-5}\}$. In addition, let $X_2 = \{0, a_1, a_2, \cdots, a_{4l+2}\}$, $X_3 = \{0, b_1, b_2, \cdots, b_{4l+2}\}$, where X_1, X_2 and X_3 have exactly one element in common, namely 0; see Figure 3.1. Since the design (X, T_1) is 2–colorable, let the vertices of X_1 be colored with 1 and 2 respectively, and let the colors of the vertices of $X_2 \cup X_3 \setminus \{0\}$ be defined as follows:

$$\phi(a_i) = \phi(b_i) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 2, & \text{otherwise.} \end{cases}$$

Therefore a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n can be obtained by letting $T = T_1 \cup T_2$ where $T_2 = \{(a_i, c_j, a_{i+1}, c_{j+1}), (b_i, c_j, b_{i+1}, c_{j+1}) \mid i = 1, 3, 5, \cdots, 4l+1; j = 1, 3, 5, \cdots, n-8l-6\} \cup \{(a_i, b_h, a_{i+1}, b_{h+1}) \mid i, h = 1, 3, 5, \cdots, 4l+1\}$. The 4–cycles in T_2 are depicted in Figure 3.1. □

Lemma 3.5. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k+5$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. Since $\binom{8k+5}{2} - i\binom{4l+3}{2}$ is not a multiple of 4 for $i = 0, 1$, a $\{C_4, K_{4l+3}\}$–design must contain at least two blocks of size $4l+3$. Therefore $n \geq 2(4l+3)-1$. Direct counting shows that $[n-(4l+2)]-(4l+2) \equiv 1 \pmod 8$. By Lemma 3.1, there exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n-(4l+2)$. By a construction similar to that shown in Figure 3.1, but adding only one block of size $4l+3$ this time, we obtain a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n. □

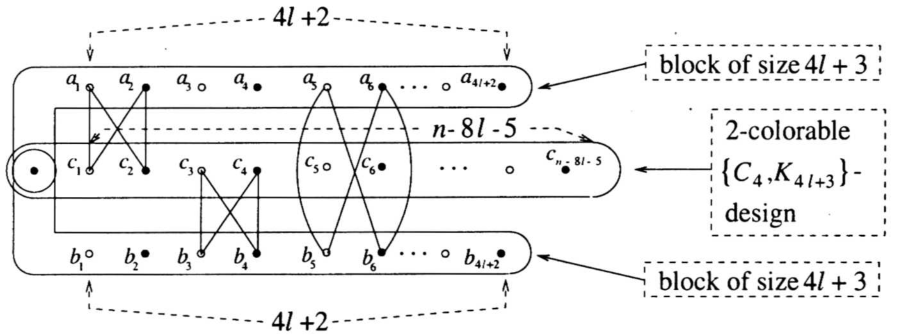


Figure 3.1: The construction of Lemma 3.4.

Lemma 3.6. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k + 7$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. If l is odd, then $n - (4l + 2) \equiv 1 \pmod 8$ and the proof follows by Lemma 3.1. On the other hand, if l is even, then $\binom{8k+7}{2} - i\binom{4l+3}{2}$ is not a multiple of 4 for $i = 0, 1$ and 2. Therefore $n \geq 3(4l + 3) - 2$. Since $[n - 2(4l + 2)] - (4l + 2) \equiv 1 \pmod 8$, a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n - 2(4l + 2)$ exists, by Lemma 3.1. Again by a construction similar to that shown in Figure 3.1, we can construct a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n directly. □

Combining Lemmas 3.2–3.6, we have the following result.

Theorem 3.7. *For each n and h, if n is an admissible order of a $\{C_4, K_{2h+1}\}$–design, then there exists a 2–colorable $\{C_4, K_{2h+1}\}$–design of order n.*

Corollary 3.8. *If there exists a 2–colorable $\{C_4, K_{2h+1}\}$–design of order n, then there exists a 2–colorable $\{C_4, C_{2h+1}\}$–design of order n.*

Proof. It is well–known that a complete graph of order $2h + 1$ can be decomposed into h hamiltonian cycles. Therefore, by replacing each K_{2h+1} with h copies of C_{2h+1}, we have the desired 2–colorable $\{C_4, C_{2h+1}\}$–design. □

Corollary 3.9. *If there exists a 2–colorable $\{C_4, K_{4l+1}\}$–design with balanced coloring, then there also exists a 2–colorable $\{C_4, C_{4l+1}\}$–design with balanced coloring.*

4. 2–Colorable $\{C_4, C_k\}$–Designs

By Corollary 3.8, we have dealt with the case of odd k. If k is a multiple of 4, a straightforward counting argument shows that a $\{C_4, C_k\}$–design must be of order $n \equiv 1 \pmod 8$. Therefore we can handle this case with a 2–colorable 4CS(n), without using any C_k.

If we insist on having a C_k in the design, then we can use the construction indicated in Figure 4.1 to obtain such a design, where we replace the 4–cycles in the shaded area with $k/4$ copies of C_k. This construction depends on Sotteau's theorem [**5**].

Theorem 4.1. [**5**] *$K_{m,n}$ can be decomposed into copies of C_{2t} if and only if m, n are even, $m, n \geq 2t$ and $2t$ divides mn.*

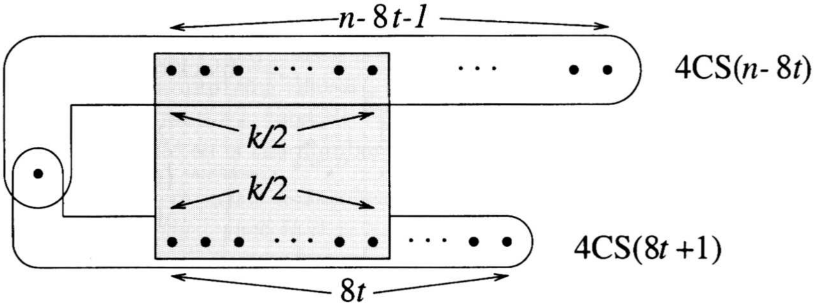


Figure 4.1: Construction of a $\{C_4, C_k\}$–design of order $n \equiv 1 \pmod 8$.

Thus $K_{k/2,k/2}$ (represented by the shaded area in Figure 4.1) can be decomposed into cycles of length k. Since each k–cycle is a hamiltonian cycle in $K_{k/2,k/2}$, it is 2–colored in accordance with the 2–colorings of the 4CS$(n-8t)$ and 4CS$(8t+1)$ in Figure 4.1.

Thus we need only consider $k \equiv 2 \pmod 4$. Now $n \equiv 1$ or $5 \pmod 8$; again a straightforward counting argument shows that we need only consider $n \equiv 5 \pmod 8$. Since $k/2$ is odd, we have to modify Figure 4.1 to make sure that we have C_k in the design. First, we need a lemma.

Lemma 4.2. *Let h be an integer such that $8h+5 > k = 4t+2$. Then $K_{8h+5} \setminus C_k$ can be decomposed into 4–cycles, and colored so that each of the 4–cycles is 2–colored. Further, the C_k is also 2–colored.*

Proof. The proof is by induction on the order $8h+5$ and on k.

First, we claim that $K_{8h+5} \setminus C_6$ can be decomposed into 2–colored 4–cycles if $8h+5 \geq 6$. For we have a 2–colored MP4CS$(8h+5)$ with leave $(0,a,c) \cup (0,b,d)$ by (ii) of Theorem 2.1. Also in the construction $(a,1,b,2)$ and $(c,1,d,2)$ are two 2–colored 4–cycles in the maximum packing. Since $(0,a,c) \cup (0,b,d) \cup (a,1,b,2) \cup (c,1,d,2)$ contains the same set of edges as $(1,a,c,2,b,d) \cup (1,c,0,b) \cup (2,a,0,d)$, and since both of $(1,c,0,b)$ and $(2,a,0,d)$ are 2–colored, our first claim is proved. The fact that the C_6 is 2–colored follows from the fact that $V(C_6) \supseteq \{a,1,b,2\}$.

Secondly, we claim that $K_{8h+5} \setminus C_{10}$ can be decomposed into 2–colored 4–cycles. To see this, let $h = h' + h'' + 1$, so that $8h+5 = (8h'+6)+(8h''+6)+1$. We already have a 2–colored MP4CS$(8h'+7)$ and a MP4CS$(8h''+7)$, each with leave a C_5. (Note here that the proof of Theorem 2.1 shows that the leave C_5 is also 2–colored. Thus there are two adjacent vertices in C_5 which have different colours; let them be d and e (d' and e' respectively) in Figure 4.2. Also let (d,d',e,e') be one of the 4–cycles between A and B, just as we have assumed in the preceding lemmas.) Now the 10–cycle can be obtained from $(a,b,c,d,e) \cup (a',b',c',d',e') \cup (d,d',e,e')$ which contains the same set of edges as $(a,b,c,d,d',c',b',a',e',e) \cup (d,e,d',e')$.

Finally we assume as our induction hypothesis that $8h+5 > 4t'+2$ and that there is a 2–colored 4–cycle decomposition of $K_{8(h-1)+5} \setminus C_{4t'+2}$, where $C_{4t'+2}$ is itself 2–colored. We claim that there is also a 2–colored 4–cycle decomposition of $K_{8h+5} \setminus C_{4t'+10}$. Since $K_9 \setminus C_8$ has a 2–colored 4–cycle decomposition, as shown in Figure 4.3, we can use the same idea again, as shown in Figure 4.4, to obtain the required construction. Since $V(C_{4t'+10}i) \supseteq V(C_{4t'+2})$, $C_{4t'+10}$ is also 2–colored.

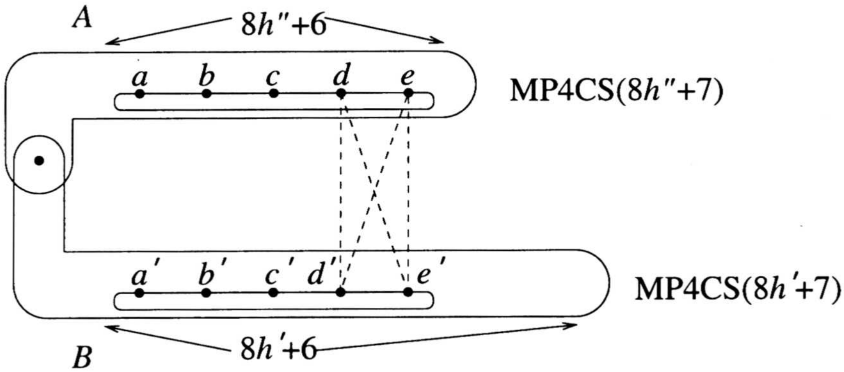


Figure 4.2: Construction of Lemma 4.1.

This completes the proof. □

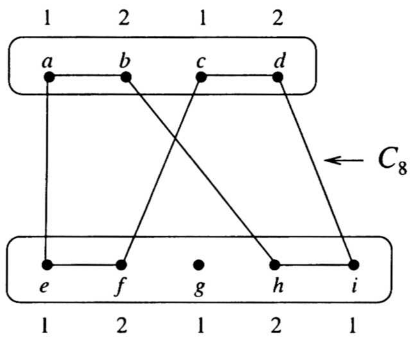


Figure 4.3: 4–cycles in $K_9 \setminus C_8$:
(a,c,b,d), (e,h,f,i), (b,e,g,f), (c,h,g,i), (a,f,d,h), (a,g,b,i), (c,e,d,g).

Theorem 4.3. *For each $k \geq 3$, if n is an admissible order of a $\{C_4, C_k\}$–design, then there exists a 2–colorable $\{C_4, C_k\}$–design of order n.*

Proof. First, if $k \equiv 0 \pmod 4$, then $n \equiv 1 \pmod 8$ and $n \geq k$. The proof then follows from the comment before Lemma 4.2.

Next, if $k \equiv 2 \pmod 4$, then $n \equiv 1$ or $5 \pmod 8$; we need only consider the case where $n \equiv 5 \pmod 8$, and the proof follows from Lemma 4.2.

Now if k is odd, then n can be any odd integer, but again we need only consider the case where $n \not\equiv 1 \pmod 8$.

1. If $k \equiv 3 \pmod 4$, the proof follows from Lemmas 3.4, 3.5, 3.6 and Corollary 3.8.

2. Finally, consider the case where $k \equiv 1 \pmod 4$.

   (a) If $n \equiv 5 \pmod 8$, the proof follows from Lemma 3.3 and Corollary 3.8.

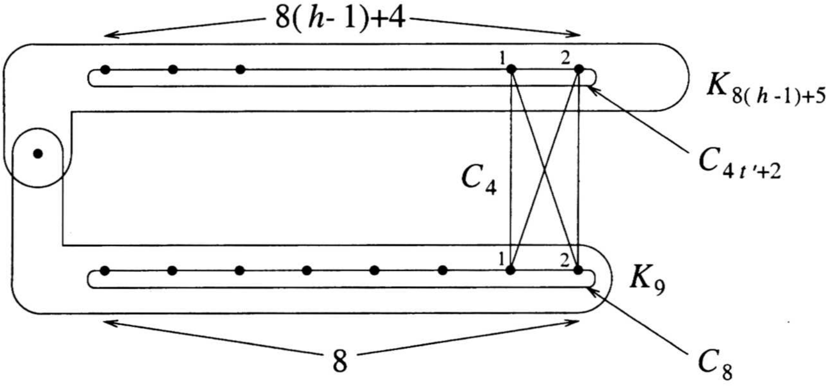


Figure 4.4: Replacing $C_{4t'+2} \cup C_4 \cup C_8$ with $C_{4t'+10} \cup C_4$.

(b) This leaves the cases where $n \equiv 3$ or $7 \pmod 8$).
By Lemmas 3.4 and 3.6, we have a 2–colorable $\{C_4, K_{4l+3}\}$–design and, by Corollary 3.9, the vertices in $V(K_{4l+3})$ have a balanced coloring, with at least $2l+1$ vertices of each colour.
Now a 2–colored $\{C_4, C_{4l+1}\}$–design exists provided that there exists a 2–colored $\{C_4, C_{4l+1}\}$–design of order $4l+3$.
In the 2–colorable $\{C_4, K_{4l+3}\}$–design that we already have, let $V(K_{4l+3}) = \{\infty_1, \infty_2\} \cup \mathbb{Z}_{4l+1}$, where ∞_i is colored i, for $i = 1, 2$. Now K_{4l+1} can be decomposed into $2l$ hamiltonian cycles and one of these, say $(0, 1, 2, \cdots, 4l)$, can be matched with ∞_1, ∞_2 to form $3l+1$ 4–cycles, as follows:

$$(\infty_1, \infty_2, 0, 4l), (\infty_1, 0, 1, 2), (\infty_2, 2, 3, 4), \cdots, (\infty_2, 4l-2, 4l-1, 4l),$$

$$(\infty_1, 1, \infty_2, 3), (\infty_1, 5, \infty_2, 7), \cdots, (\infty_1, 4l-3, \infty_2, 4l-1).$$

Since we can arrange the colours of $0, 1, 2, \cdots, 4l$, to alternate between 1 and 2, all these 4–cycles are 2–colored, and K_{4l+3} is now decomposed into $2l-1$ cycles of length $4l+1$ and $3l+1$ 4-cycles, all of which are 2–colored.
This completes the proof. □

5. Concluding Remarks

We have constructed 2–colorable $\{C_4, K_{2h+1}\}$–designs in Section 3. This suggests that a 2–colorable $\{C_4, K_k\}$–design may well exist for each $k \geq 3$. The construction of such a design is easy for $k = 4$ and $k = 8$, but we have been unable to construct one for $k = 6$.

We recall that Alspach [**1**] asked the following question in 1981: Let n be a positive integer and let $a_1 + a_2 + \cdots + a_r$ be a partition of either $\binom{n}{2}$ if n is odd, or $\binom{n}{2} - n/2$ if n is even, such that $3 \leq a_i \leq n$ for $i = 1, 2, \cdots, r$. Does there exist a partition, into cycles of lengths $a_1, a_2, \cdots, a_r$, of the edge–set of K_n when n is odd, or of K_n with a 1–factor removed when n is even?

The existence of a $\{C_4, C_k\}$–design for all admissible orders certainly suggests that Alspach's conjecture may hold for two cycle sizes, 4 and k. To construct a 2–colorable $\{C_4, C_k\}$-design with a prescribed number of 4–cycles (and hence a prescribed number of k–cycles) sounds feasible and interesting. Note that for $k = 3$, some restriction must be made; for instance the 2–colorable $\{C_3, C_4\}$–design cannot have too many triangles [**3**].

Acknowledgement. This result was obtained while the first and second authors were visiting the Centre for Discrete Mathematics and Computing, The University of Queensland. They thank the third author for her invitation.

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