

THE INTERSECTION PROBLEM OF LATIN SQUARES

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22

The intersection problem of latin squares was first studied around ten years ago when T. Webb considered the intersections of two idempotent commutative latin squares. Since then, a few results have been obtained on this problem. In this paper, we will survey all the results we have got so far and some of their applications will also be mentioned.

1. INTRODUCTION

A latin square of order v is a $v \times v$ array such that each of the integers $1, 2, 3, \dots, v$ (or any set of v distinct symbols) occurs exactly once in each row and each column. Two latin squares of the same order, $L = [l_{i,j}]$ and $M = [m_{i,j}]$, are said to have intersection k , denoted by $|L \cap M| = k$, if there are exactly k cells (i, j) such that $l_{i,j} = m_{i,j}$. The intersection of $t (> 2)$ distinct latin squares of the same order can be defined similarly. The intersection problems mainly study for which integer $k \in \{0, 1, 2, \dots, v^2\}$ there exist two (or t) latin squares (of certain type) which have intersection k . So far, the intersections of many types of latin squares have been studied. Table 1.1 is a list of them. For clearness, we give their definitions

TABLE 1.1

Type	Latin Square	Shorthand
1	—	LS
2	Idempotent	ILS
3	Unipotent	ULS
4	Idempotent Commutative	ICLS
5	Commutative	CLS
6	Unipotent Commutative	UCLS
7	Commutative, with Holes	CLSH
8	Half-Idempotent	HILS

in what follows. A latin square $L = [l_{i,j}]$ is idempotent if $l_{i,j} = i$ for each i and it is **unipotent** provided that $l_{i,j} = c$ for each i and a fixed c . We say L is **commutative** if $l_{i,j} = l_{j,i}$ for all i and j . A latin square of even order n which contains $n/2$ latin subsquares based on $\{x, y\}$ in the cells $\{x, y\} \times \{x, y\}$, where $\{x, y\} \in H = \{\{1, 2\}, \{3, 4\}, \dots, \{n-1, n\}\}$, is called a **latin square with filled holes** of size 2. It is a well-known result that a commutative latin square of order n with filled holes of size 2 exists for each even $n \geq 6$, [3]. Finally, a latin square $L = [l_{i,j}]$ of order $2n$ is said to be **half-idempotent** if $l_{i,j} = i$ whenever $1 \leq i \leq n$ and $l_{i,j} = i - n$ whenever $n + 1 \leq i \leq 2n$.

2. THE BASIC TECHNIQUES

Since the intersections of $t (> 2)$ distinct latin squares can be found by a similar way as the intersections of two latin squares, hence in this paper, we consider the intersections of two latin squares only. Let L and M be two latin squares (of certain type) which have intersection k . By taking away all the common entries, we obtain two partial latin squares which satisfy the following conditions:

- (1) The corresponding cells in these two partial latin squares are either both filled or both empty.
- (2) The entries in the corresponding filled cells are distinct.
- (3) To each row and each column, these two partial latin squares contain the same set of entries.

The above conditions suggest us the following definitions. Two partial latin squares are said to be **comparable** if (1) holds, two partial latin squares are **disjoint** if (1) and (2) hold, and two partial latin squares are called **mutually balanced** provided (1) and (3) hold.

Thus, when we are looking for possible intersections k of two latin squares of order v , we have to check first the existence of two disjoint and mutually balanced (DMB) partial latin squares with $v^2 - k$ cells filled. As an example, since there don't exist two DMB partial latin squares with either 1 or 2 or 3 or 5 cells filled, the intersections $v^2 - 5$, $v^2 - 3$, $v^2 - 2$, and $v^2 - 1$ are simply not possible. That is, if we define $J_i[v]$ to be the set of all intersections between two latin squares of order v and of type i , then we have the following result.

PROPOSITION 2.1.

$$J_i[v] \subseteq I_1[v] = \{0, 1, 2, \dots, v^2\} \setminus \{v^2 - 5, v^2 - 3, v^2 - 2, v^2 - 1\}, i = 1, 2, \dots, 8.$$

On the above proposition, if $i = 1$, the set $I_1[v]$ is quite good for the possible intersections, as a matter of fact $J_1[v] = I_1[v]$ for each $v \geq 5$, [9]. But, for the other types of latin squares, $I_1[v]$ may be too large. Simply

consider the intersections of two idempotent latin squares, we know $0, 1, 2, \dots, v-1$ are not possible intersections at all. Thus we have to determine the set of possible intersections of latin squares of type i , denoted by $I_i[v]$, then prove the following proposition first. (We omit the proof.)

PROPOSITION 2.2. $J_i[v] \subseteq I_i[v], i = 1, 2, \dots, 8.$

Next, we have to determine exactly which intersection can actually be obtained. In another words, we need to construct a pair of latin squares of order v and type i which have the intersection we expect. In mathematical term, we want to show that $J_i[v] \supseteq I_i[v]$ for $v \geq c$, c is a fixed small integer. Of course, we will not do this job by constructing a pair of latin squares of order v and type i with intersection k for each v, i , and $k \in I_i[v]$ independently. Before we go any further we need the following definitions.

A (partial) latin square of order u , $P = [p_{i,j}]$, is said to be embedded in a latin square of order v , if there exists a latin square $L = [l_{i,j}]$ of order v such that $l_{i,j} = p_{i,j}$ for each filled cell in P . The embedding of a (partial) latin square of certain type can be defined similarly. For the results of embedding the readers can refer to [1, 2]. As an example, a (partial) latin square of order u can be embedded in a latin square of order $v \geq 2u$.

With the above embedding results, we can easily develop some recursive constructions. In what follows we will use idempotent latin squares to describe the techniques. We first establish two recursive constructions by using the embedding results.

PROPOSITION 2.3. *If $J_2[v] = I_2[v], v \geq 5$, then $J_2[2v+1] = I_2[2v+1]$ and $J_2[2v+2] = I_2[2v+2]$, [9].*

Proof. We outline the proof. Since an $ILS(v)$, A , can be embedded in an $ILS(2v+1)$, by using $J_2[v] = I_2[v]$ and the permutation of the entries outside A , we obtain $J_2[2v+1] = I_2[2v+1]$. The other case can be obtained similarly.

After we got these recurrence relations, all we need are the ingredients, $J_2[v], v \leq 2t$, if $J_2[t] = I_2[t]$ for some small t . Since, by direct constructions, we have $J_2[6] = I_2[6]$. Thus, we have to show $J_2[7] = I_2[7]$, $J_2[8] = I_2[8], \dots$, and $J_2[12] = I_2[12]$ by constructing a pair of idempotent latin squares for each possible intersection. It was a very difficult job around ten years ago. But, it is easier now; with the help of computer we can first generate a list of distinct latin squares of certain type (of small order), then the intersection can be obtained right away. This idea can be seen in [4, 5]. The idea can also be utilized to find all the possible intersections for latin squares of very small order.

We remark here that the recursive constructions of Proposition 2.3 will be changed if we consider different type of latin square.

As we have mentioned above, in order to see whether h is a possible intersection of two latin squares of certain type, we first check the existence of two DMB partial latin squares with $v^2 - h$ cells filled. But even if we can find DMB partial latin squares, we still cannot conclude that h is a possible intersection. Figure 2.1 gives an example that there exist two DMB partial latin squares of order 3 with 7 cells filled, but 2 is not a possible intersection of latin squares of order 3. The reason is very

1	2	
2	1	3
	3	2

2	1	
1	3	2
	2	3

Fig. 2.1

simple. We have to make sure that these DMB partial latin squares of order 3 can be completed to latin squares of order 3. No way we can do that. From this fact we can easily see that some possible intersections are not the intersections of small order latin squares. Thus the intersection problem has come to determine the smallest (nontrivial) order t of certain type of latin squares in which $J_i[t] = I_i[t]$. And it is believed that for each admissible order $v \geq t$ we are going to have $J_i[v] = I_i[v]$. This observation works for almost all different types of latin squares so far have been studied except the intersections of two commutative latin squares. Due to the fact that a commutative latin square of odd order v must be diagonal, i.e., $\{I_i, i: 1 \leq i \leq v\} = \{1, 2, \dots, v\}$, $v^2 - 7$ and $v^2 - 4$ are not possible intersections, [6]. But, these two numbers are possible intersections for commutative latin squares of even order. Thus, we have to consider odd order and even order separately. One more fact is worthy of mention, in the case of HILS, since we can construct an HILS of order $2v$ by using the way in Fig. 2.2, hence we can obtain the intersections of HILS($2v$) by way of $J_1[v]$ and $J_2[v]$. Thus no recursive construction is necessary. Similarly, the intersections of two CLSH($2v + 2$) can be obtained by the other way [7], all we need is one recursive construction.

HILS ($2v$) :	A ILS(v)	B LS(v)	A, D based on $\{1, 2, \dots, v\}$
	C LS(v)	D ILS(v)	

Fig. 2.2

For clearness, we use Table 2.3 to describe all results we know in the intersection problem of latin squares, and omit the details.

Type	Possible Intersections	Recursive Construction	Admissible Orders	Smallest Order	Reference
1. LS	$I_1[v]$	$v \rightarrow 2v$ $v \rightarrow 2v + 1$	All	6	[9]
2. ILS	$*I_2[v]$	$v \rightarrow 2v + 1$ $v \rightarrow 2v + 2$	All	6	[9]
3. ULS	$I_2[v]$	$v \rightarrow 2v$ $v \rightarrow 2v + 1$	All	6	[9]
4. ICLS	$\dagger I_2[v]$	$v \rightarrow 2v + 1$ $v \rightarrow 2v + 3$	Odd	7	[22]
5. CLS	$\left\{ \begin{array}{l} \text{Odd: } I_1[v] \setminus \\ \{v^2 - 7, v^2 - 4\} \\ \dots\dots\dots \\ \text{Even: } I_1[v] \end{array} \right.$	$v \rightarrow 2v + 1$ $v \rightarrow 2v + 3$ $\dots\dots\dots$ $v \rightarrow 2v$ $v \rightarrow 2v + 2$	All	7 ...	[6]
6. UCLS	$I_3[v]$	$v \rightarrow 2v$ $v \rightarrow 2v + 2$	Even	8	[11, 15]
7. CLSH	$\ddagger I_4[v]$	$v \rightarrow 2v + 4$	Even	10	[7]
8. HILS	$I_2[v]$	None	Even	≤ 10	[7]

$*I_2[v] = I_1[v] \setminus \{0, 2, \dots, v - 1\}$

$\dagger I_3[v] = \{v + 2k : k \in \{0, 1, 2, \dots, t_r - 7, t_r - 6, t_r - 4, t_r\}, t_r = v(v - 1)/2\}$

$\ddagger I_4[v] = \{2v + 2k : k \in \{0, 1, 2, \dots, s_r - 7, s_r - 9, s_r - 4, s_r\}, s_r = (v - 2)/2\}$

3. APPLICATIONS

It is well-known that latin squares can be used to construct special block design, for example, a latin square of order v and three disjoint Steiner triple system of order v will produce a Steiner triple system of order $3v$ by a tripling construction. Due to this fact, we can apply the intersections of latin squares to study the intersections of two designs if the designs are obtained from certain type of latin squares.

One of the earliest results is the intersection of two Steiner triple systems which is solved by C. C. Lindner and A. Rosa in [16]. If we apply the following constructions, A and B , of Steiner triple systems, then almost all the intersections of two Steiner triple systems turn out to be a direct result of the intersections of two idempotent commutative latin squares and two commutative latin squares with filled holes of size two.

CONSTRUCTION A. Let T be the set $\{1, 2, \dots, 2v\} (v \geq 3)$ and $S = (T \times \{1, 2, 3\}) \cup \{\alpha\}$. Also let L_1, L_2 , and L_3 be three $CLSH(2v)$. Define a collection of triples, t , on S as follows:

- (i) if $\{x, y\}$ belongs to the holes, define an STS(7) on the set $\{(x, i), (y, i), \infty : i = 1, 2, 3\}$; and
- (ii) if $\{x, y\}$ is not a hole, let $\{(x, i), (y, i), (z, i + 1)\} \in t$ where the entry in (x, y) of L_i is z , $i = 1, 2, 3$ and $i + 1$ takes modulo 3.

Then (S, t) is an STS($6v + 1$).

CONSTRUCTION B. Let T be the set $\{1, 2, \dots, 2v + 1\}$ and $S = T \times \{1, 2, 3\}$. Also, let L_1, L_2 and L_3 be three CILS($2v + 1$). Define a collection of triples, t , on S as follows:

- (i) $\{(x, 1), (x, 2), (x, 3)\} \in t$ for each $x \in T$; and
- (ii) $\{(x, i), (y, i), (z, i + 1)\} \in t$ if the entry in (x, y) of L_i is z , $i = 1, 2, 3$, and $i + 1$ takes modulo 3.

Then (S, t) is an STS($6v + 3$).

Similar to the above result, the intersections of two Mendelsohn triple systems and Transitive triple systems can also be obtained by way of tripling constructions and the intersections of two idempotent latin squares. This will provide the same results as in [10, 12].

More recently, the intersections of pentagon systems are also obtained by using the intersections of CLSH[3]. We believe some intersections of the other odd cycle systems [20, 21] can be found similarly.

The intersections of latin squares can also be used to find the intersections of some other special type of triple systems, we just cannot include them all here. One more fact we would like to mention is that a 1-factorization is equivalent to a unipotent commutative latin square, and the intersections of two 1-factorizations act a very important role in obtaining the intersections of two Steiner quadruple systems of order $2v$, [19].

Recently, the intersections of more than two latin squares and two designs have been studied and applied [4, 5, 7, 17, 18, 19]. We expect more applications will be found in the near future, and this is why we try to put the above known results together and hopefully bring this topic into the attention of more people.

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