Almost Resolvable Directed 2r-Cycle Systems

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ABSTRACT

In this article it is shown that there exists a directed $2r$-cycle system of D_n, the complete directed graph, if and only if $n \equiv 1 \pmod{2r}$. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

An m-cycle $(v_0, v_1, \ldots, v_{m-1})$ is a graph with vertex set $\{v_i | i \in \mathbb{Z}_m\}$ and edge set $\{v_i v_{i+1} | i \in \mathbb{Z}_m\}$, reducing the subscript modulo m. An m-cycle system of *order* n and *index* λ is an ordered pair (V, C) where $|V| = n$ and C is a set of m-cycles that partition

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the edges of λK_n (the multigraph in which each pair of vertices is joined by λ edges) defined on the vertex set V. A *parallel class* of an m-cycle system (V, C) of order n is a set of n/m vertex disjoint m-cycles in C, and an *almost parallel class* is a set of $(n - 1)/m$ vertex *disjoint* m-cycles in C. The vertex of V that is in no m-cycle of an almost parallel class π of C is said to be the vertex *missing* from π. An m-cycle system (V, C) is said to be *(almost) resolvable* if C can be partitioned into (almost) parallel classes.

There has been much interest over the past 30 years in the problem of the existence of m-cycle systems, and those satisfying additional properties. While much is known about the existence of m-cycle systems (see [7] for a survey), it is unlikely that the problem will be solved in the near future. On the other hand, the existence of resolvable m-cycle systems of K_n has been completely solved [1], as has the related problem of finding resolvable m-cycle systems of $K_n - F$, where F is a 1-factor of K_n [1,6]. Similarly, the existence of almost resolvable m-cycle systems of λK_n has also been solved. In this case, it is necessary that λ be even and that $n \equiv 1 \pmod{m}$, and these conditions are also sufficient [3,5].

A natural companion problem to those just presented is to ask the same question for directed graphs. A directed m-cycle $(v_0, v_1, \ldots, v_{m-1})$ is a directed graph with vertex set $\{v_i | i \in \mathbb{Z}_m\}$ and set of arcs $\{(v_i, v_{i+1}) | i \in \mathbb{Z}_m\}$, reducing the subscript modulo m. A directed m-cycle system of order n is an ordered pair (V, C) where C is a set of directed m-cycles that partition the arcs of D_n (the complete directed (symmetric) graph) with vertex set V.

The existence of resolvable directed m-cycle systems of order $n \equiv m \pmod{2m}$ can be established by taking two copies of each m-cycle in a solution for the undirected case and directing the copies in opposite directions. However this solution technique cannot be used to settle the existence of almost resolvable directed m-cycle systems, since in the undirected case λ must be even. One could try to direct the cycles defined in [3,5], but this is not always possible. If m is odd then in [5] the construction requires the use of skew Room frames and, in order for these cycles to be directed, an additional property is required of the skew Room frames. Obtaining almost resolvable directed m-cycle systems using this approach has been used to settle the problem for the case $m = 5$ [4]. On the other hand, if m is even then, for example, the m-cycles in the almost resolvable m-cycle systems of $2K_{2m+1}$ defined in [3] can never be directed to form an almost resolvable directed m-cycle system of D_{2m+1}. In this article, we provide a neat new construction that proves Theorem 2.7, settling the existence of almost resolvable directed $2r$-cycle systems of all orders. As opposed to the construction in [3] where the existence of an almost resolvable m-cycle system of $2K_{2m+1}$ was an essential ingredient in establishing the existence for all orders, the construction defined here needs only an almost resolvable m-cycle system of D_{m+1}. Theorem 2.7 answers an open problem stated in [7]. Throughout the rest of this article we assume that $m = 2r$, so is even.

2. THE CONSTRUCTION

Before presenting the main theorem we need some preliminary results. Let $D_{x,y}$ be the complete (symmetric) directed bipartite graph.

Theorem 2.1 ([8]). *There exists a directed 2r-cycle system of* $D_{x,y}$ *if and only if r divides xy and* $x, y \geq r$.

From this we can easily obtain the following corollary.

Corollary 2.2. *There exists a resolvable directed 2r-cycle system of* $D_{x,y}$ *if and only if* $x = y \equiv 0 \pmod{r}$.

Proof. The necessity is clear, so consider $D_{pr,pr}$ with bipartition of the vertex set $\mathbb{Z}_p \times \mathbb{Z}_r \times \{0\}$ and $\mathbb{Z}_p \times \mathbb{Z}_r \times \{1\}$. Let $F_0, \ldots, F_{p-1}$ be p 1-factors that form a 1-factorization of $K_{p,p}$ with bipartition of the vertex set $\mathbb{Z}_p \times \{0\}$ and $\mathbb{Z}_p \times \{1\}$. For each $i, j \in \mathbb{Z}_p$, let $\{c_{i,j,0}, \ldots, c_{i,j,r-1}\}$ be a set of directed $2r$-cycles that form a directed $2r$-cycle system of $D_{r,r}$ with bipartition of the vertex set $\{i\} \times \mathbb{Z}_r \times \{0\}$ and $\{j\} \times \mathbb{Z}_r \times \{1\}$ (see Theorem 2.1). For each $q \in \mathbb{Z}_p$ and each $k \in \mathbb{Z}_r$, let

$$C_{q,k} = \{c_{i,j,k} | \{(i,0),(j,1)\} \in F_q\},$$

so $C_{q,k}$ is a parallel class of $D_{pr,pr}$. Since $(\mathbb{Z}_p \times \mathbb{Z}_r \times \mathbb{Z}_2, \{c \mid c \in C_{q,k}, q \in \mathbb{Z}_p, k \in \mathbb{Z}_r\})$ is easily seen to be a directed $2r$-cycle system of $D_{pq,pq}$, the result follows. □

We need one more ingredient. A *symmetric* latin square with holes of size 2 on the symbols $1, \ldots, 2n$ is a symmetric latin square in which for $1 \leq i \leq n$ cells $\{2i-1, 2i\} \times \{2i-1, 2i\}$ form a subsquare on the symbols $2i-1$ and $2i$. The following is well known.

Lemma 2.3. *There exists a symmetric latin square with holes of size 2 and of order 2x for all* $x \geq 3$.

Corollary 2.4. *Let F be a 1-factor of* K_{2x}. *For all* $x \geq 3$ *there exists a 1-factorization of the multigraph* $K_{2x} + F$ *in which the 2x edges in 2F occur in different 1-factors.*

Proof. Form an array T from a symmetric latin square with holes of size 2 on the symbols $1, \ldots, 2x$ by removing the symbols in diagonal cells, and by having both the two symbols $2i-1$ and $2i$ occur in both cells $(2i-1, 2i)$ and $(2i, 2i-1)$ for $1 \leq i \leq x$. For $1 \leq i \leq 2x$ define

$$F_i = \{\{a,b\} \mid \text{cell } \{a,b\} \text{ of } T \text{ contains symbol } i\}.$$

Since T is symmetric and the diagonal is empty, and since each symbol occurs exactly once in each row and each column of T it is clear that $|F_i| = x$, that F_i is a 1-factor of K_{2x}, and that $\{F_i \mid 1 \leq i \leq 2x\}$ is a 1-factorization of $K_{2x} + F$, where $F = \{\{2i-1, 2i\} \mid 1 \leq i \leq x\}$. Furthermore, $\{2i-1, 2i\} \in F_{2i-1}, F_{2i}$, for $1 \leq i \leq x$, so the edges in F occur in different 1-factors as required. □

Our search for almost resolvable directed m-cycle systems of D_n begins with the first case, $n = m + 1$. It is well known that there exists a directed m-cycle system of D_{m+1} (use a standard difference method construction), and clearly it is almost resolvable as each directed m-cycle misses exactly one vertex, so we have the following.

Lemma 2.5 ([2]). *There exists an almost resolvable directed m-cycle system of* D_{m+1}.

We can also find an almost resolvable $2r$-cycle system of D_{4r+1}.

Lemma 2.6. *There exists an almost resolvable directed 2r-cycle system of D_{4r+1} for all $r \geq 2$.*

Proof. Let $c = (c_1, c_2, \ldots, c_{2r})$ where

$$c_i = \begin{cases} (-1)^{i+1} i \bmod (4r+1) & \text{for } 1 \leq i \leq r-1, \\ (-1)^i i \pmod{4r+1} & \text{for } r \leq i \leq 2r. \end{cases}$$

Let $-c = (-c_1, -c_2, \ldots, -c_{2r})$ (again, reducing modulo $4r + 1$).

Then $(\mathbb{Z}_{4r+1}, \{c + i, -c + i \mid i \in \mathbb{Z}_{4r+1}\})$ is an almost resolvable directed $2r$-cycle system, with almost parallel classes $\pi_i = \{c + i, -c + i \mid i \in \mathbb{Z}_{4r+1}\}$ (π_i has deficiency i). □

Finally, we can present the main result.

Theorem 2.7. *There exists an almost resolvable directed 2r-cycle system of D_n if and only if $n \equiv 1$ (mod m).*

Proof. Since the number of vertices in an almost parallel class of D_n is $n - 1$, it is clear that m divides $n - 1$ so the necessity follows. To prove the sufficiency, let $n = xm + 1$; by Lemmas 2.5 and 2.6 we can assume that $x \geq 3$.

Let $F = \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_x\}$. Let $F_0, F_1, \ldots, F_{2x-1}$ be a 1-factorization of $K_{2x} + F$ in which the edge $\{2i, 2i+1\}$ occurs in F_{2i} and in F_{2i+1} for each $i \in \mathbb{Z}_x$ (see Corollary 2.4). For each $i \in \mathbb{Z}_x$ let $(\{\infty\} \cup (\{2i, 2i+1\} \times \mathbb{Z}_{m/2}), C_i)$ be an almost resolvable directed m-cycle system of D_{m+1} (see Lemma 2.5). For each $i \in \mathbb{Z}_{2x}$ and each $j \in \mathbb{Z}_{m/2}$, let $c_{i,j} \in C_{\lfloor i/2 \rfloor}$ be the directed m-cycle that misses the vertex (i, j), and for each $l \in \mathbb{Z}_x$ let $c_l \in C_l$ be the directed m-cycle that misses the vertex ∞. So for each $i \in \mathbb{Z}_x$, $C_i = \{c_{2i,j}, c_{2i+1,j} \mid j \in \mathbb{Z}_{m/2}\} \cup \{c_i\}$. Finally, for each $i \in \mathbb{Z}_{2x}$ and each $\{a, b\} \in F_i \backslash \{2\lfloor i/2 \rfloor, 2\lfloor i/2 \rfloor + 1\}$, let $(\{a, b\} \times \mathbb{Z}_{m/2}, \{c_{a,b,0}, c_{a,b,1}, \ldots, c_{a,b,m/2-1}\})$ be a (resolvable) directed m-cycle system of $D_{m/2,m/2}$ with bipartition of the vertex set $\{a\} \times \mathbb{Z}_{m/2}$ and $\{b\} \times \mathbb{Z}_{m/2}$ (see Theorem 2.1).

We now have all the ingredients to describe the almost resolvable m-cycle system of D_{xm+1} with vertex set $\{\infty\} \cup (\mathbb{Z}_{2x} \times \mathbb{Z}_{m/2})$. For each $i \in \mathbb{Z}_{2x}$ and each $j \in \mathbb{Z}_{m/2}$, let

$$\pi_{i,j} = \{c_{a,b,j} \mid \{a, b\} \in F_i \backslash \{2\lfloor i/2 \rfloor, 2\lfloor i/2 \rfloor + 1\}\} \cup \{c_{i,j}\}, \text{ and}$$
$$\pi_\infty = \{c_l \mid l \in \mathbb{Z}_x\},$$

and let $C = (\bigcup_{i \in \mathbb{Z}_{2x}, j \in \mathbb{Z}_{m/2}} \pi_{i,j}) \cup \pi_\infty$ be a set of m-cycles defined on the vertex set $\{\infty\} \cup (\mathbb{Z}_{2x} \times \mathbb{Z}_{m/2})$. Then $\pi_{i,j}$ is an almost parallel class of C that is missing the vertex (i, j), and π_∞ is an almost parallel class of C that is missing the vertex ∞. Also,

(a) each arc $((a, y), (b, z))$ with $\{a, b\} \subseteq \{2i, 2i+1\}$ for some $i \in \mathbb{Z}_x$ (possibly $a = b$), is in $c_{2i,j}$ or $c_{2i+1,j}$ for some $j \in \mathbb{Z}_{m/2}$, or is in c_i,

(b) each arc $((a, y), \infty)$ with $a \in \{2i, 2i+1\}$, for some $i \in \mathbb{Z}_x$, is in $c_{2i,j}$ or $c_{2i+1,j}$ for some $j \in \mathbb{Z}_{m/2}$, and

(c) each arc $((a, y), (b, z))$ with $\{a, b\} \not\subseteq \{2i, 2i+1\}$ for any $i \in \mathbb{Z}_x$ is in $c_{a,b,j}$ for some $j \in \mathbb{Z}_{m/2}$.

Therefore $(\{\infty\} \cup (\mathbb{Z}_{2x} \times \mathbb{Z}_{m/2}), (\bigcup_{\substack{i\in\mathbb{Z}_{2x}\\ j\in\mathbb{Z}_{m/2}}} \pi_{i,j}) \cup \pi_\infty)$ is seen to be an almost resolvable directed m-cycle system of D_{xm+1} with almost parallel classes π_∞ and $\pi_{i,j}$ for $i \in \mathbb{Z}_{2x}$ and $j \in \mathbb{Z}_{m/2}$. □

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