SIAM J. NUMER. ANAL.
Vol. 40, No. 2, pp. 578–600

NUMERICAL SCHUBERT CALCULUS BY THE PIERI HOMOTOPY ALGORITHM*

T. Y. LI†, XIAOSHEN WANG‡, AND MENGNIEN WU§

Abstract. Based on Pieri's formula on Schubert varieties, the Pieri homotopy algorithm was first proposed by Huber, Sottile, and Sturmfels [*J. Symbolic Comput.*, 26 (1998), pp. 767–788] for numerical Schubert calculus to enumerate all p-planes in $\mathbb{C}^{m+p}$ that meet n given planes in general position. The algorithm has been improved by Huber and Verschelde [*SIAM J. Control Optim.*, 38 (2000), pp. 1265–1287] to be more intuitive and more suitable for computer implementations.

A different approach of employing the Pieri homotopy algorithm for numerical Schubert calculus is presented in this paper. A major advantage of our method is that the polynomial equations in the process are all square systems admitting the same number of equations and unknowns. Moreover, the degree of each polynomial equation is always 2, which warrants much better numerical stability when the solutions are being solved. Numerical results for a big variety of examples illustrate that a considerable advance in speed as well as much smaller storage requirements have been achieved by the resulting algorithm.

Key words. enumerative geometry, Schubert variety, Pieri formula, Pieri homotopy algorithm, Pieri poset

AMS subject classifications. 14N10, 14M15, 65H10, 68Q40

PII. S003614290139175X

1. Introduction.

With "l-planes" representing l dimensional linear subspaces, a general problem in enumerative geometry is

($\star$) Enumerate all p-planes in $\mathbb{C}^{m+p}$ that meet n given planes $L_1, \dots, L_n$ in general position of dimension $m+1-k_i$ for $i = 1, \dots, n$, with $k_1 + \cdots + k_n = mp$.

The condition that $k_1 + \cdots + k_n = mp$ guarantees a finite number of p-planes meeting those given planes.

Based on Pieri's formula, and following the new geometric proof of Pieri's formula established by Sottile [9], Huber, Sottile, and Sturmfels [3] proposed the *Pieri homotopy algorithm* to deal with this problem numerically. The homotopies in the algorithm have then been simplified by Huber and Verschelde [4] via the poset of localization patterns, making the algorithm more suitable for computer implementations.

In both of those works, each given plane L_i for $i = 1, \dots, n$ with dimension $d_i = m + 1 - k_i$ is represented, as they were traditionally, by an $(m+p) \times d_i$ matrix consisting of d_i linearly independent vectors in $\mathbb{C}^{m+p}$. Let X be a p-plane that intersects all those given planes. Without loss, one may represent X by the $(m+p) \times p$

*Received by the editors July 2, 2001; accepted for publication (in revised form) January 17, 2002; published electronically June 12, 2002.

http://www.siam.org/journals/sinum/40-2/39175.html

†Department of Mathematics, Michigan Sate University, East Lansing, MI 48824 (li@math.msu.edu). The research of this author was supported in part by NSF under grant DMS-0104009.

‡Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204 (xxwang@ualr.edu). The research of this author was supported in part by the Visiting Scholar Foundation of Ministry of Education of China. Part of this author's work was done during a stay at the Key Laboratory for Symbolic Computation and Knowledge Engineering in China.

§Department of Mathematics, Tamkang University, Taipei, Taiwan (wu@math.tku.edu.tw).

matrix

$$\begin{bmatrix} 1 & & 0 \\ x_{11} & \ddots & \\ \vdots & \ddots & 1 \\ x_{m1} & & x_{1p} \\ & \ddots & \vdots \\ 0 & & x_{mp} \end{bmatrix}.$$

For $i = 1, \ldots, n$, let $[\alpha]_i(x)$, where $x = (x_{11}, \ldots, x_{mp})$, denote the maximal minor of the $(m+p) \times (p+d_i)$ matrix $[X|L_i]$ with row indices $\alpha = (\alpha_1, \ldots, \alpha_{p+d_i})$. Then the intersection conditions in problem $(\star)$ become, for each $i = 1, \ldots, n$,

$$X \text{ meets } L_i \iff [\alpha]_i(x) = 0 \quad \forall \text{ possible row indices } \alpha = (\alpha_1, \ldots, \alpha_{p+d_i}).$$

The backbone of the Pieri homotopy algorithm [3, 4] is to solve k_i more variables in $x = (x_{11}, \ldots, x_{mp})$ one at a time for $i = 1, \ldots, n$ successively to satisfy the intersection conditions with $L_1, \ldots, L_i$:

$$(1) \qquad [\alpha]_l(x) = 0 \ \ \forall \text{ possible row indices } \alpha = (\alpha_1, \ldots, \alpha_{p+d_l}), \text{ for } l = 1, \ldots, i.$$

To solve the above systems successively for $i = 1, \ldots, n$, different homotopies based on Pieri's formula on Schubert varieties are constructed at each stage where the solutions of the system at the current stage taken as the solutions of the target system of the current homotopy are the solutions of the start system of the homotopy at the next stage. In the process, if $k_i = 1$, then $d_i = m + 1 - 1 = m$, making $[X|L_i]$ a square matrix and resulting in the increment of one more equation in one more unknown in (1) from $(i-1)$th stage to ith stage. However, when $k_i > 1$, then $d_i = m + 1 - k_i < m$, and consequently the number of all possible maximal minors in the $(m+p) \times (p+d_i)$ matrix $[X|L_i]$ equals

$$\binom{p+m}{p+d_i} = \binom{p+m}{p+m+1-k_i} = \binom{p+m}{k_i - 1} > k_i$$

since $k_i = m+1-d_i < m$. When this occurs, the system in (1) admits more equations than unknowns and constitutes an overdetermined system.

Solving an overdetermined system by the homotopy continuation method as proposed in [8], a square system is constructed by using random linear combinations of all equations in (1). This reduction to a square system destroys the geometric structure and creates many excess solution paths to follow, which may lead to a considerable inefficiency of the algorithm since the solution sets of the new square system may properly contain the original ones.

In this paper, we present a different approach. Most importantly, we will represent each given plane L_i, $i = 1, \ldots, n$, in general position by a set of $m+p-d_i = p+k_i-1$ linear equations which defines L_i. The collection of the normals of those equations forms a $(p+k_i-1) \times (m+p)$ matrix, denoted by K_i, and X meets L_i if and only if

$$K_i X \Lambda_i = 0 \quad \text{for some } \Lambda_i \in \mathbb{P}^{p-1}.$$

Employing the same strategy as in [3, 4], we will solve k_i more variables in $x = (x_{11}, \ldots, x_{mp})$ one at a time from $i = 1$ to $i = n$ by solving for each i the system

$$(2) \qquad K_l X \Lambda_l = 0 \ \ \Lambda_l \in \mathbb{P}^{p-1} \qquad \text{for } l = 1, \ldots, i.$$

And different homotopies are constructed, also based on Pieri's formula, at different stages to connect the solutions of the systems in (2) for consecutive i's. For each fixed i, the system in (2) has $(p+k_1-1)+\cdots+(p+k_i-1)=(p-1)i+k_1+\cdots+k_i$ equations. On the other hand, since $\Lambda_l \in \mathbb{P}^{p-1}$, it admits only $p-1$ variables for each l; together with $k_1+\cdots+k_i$ variables in $x=(x_{11},\ldots,x_{mp})$, the system has $(p+k_1-1)+\cdots+(p+k_i-1)=(p-1)i+k_1+\cdots+k_i$ variables. We therefore deal with square systems throughout the process even when $k_i>1$ occurs and never have to undertake the disadvantages of solving overdetermined systems. Moreover, another important advantage of our approach is that the degree of each polynomial equation in (2) is always 2 while polynomial equations in the previous approaches in [3, 4] may reach quite higher degrees in many situations, which may severely affect the numerical stability when solutions of the systems are being solved.

The computational experiences of the resulting algorithm are listed at the end of the paper to illustrate the remarkable speed up of our method has achieved over the existing algorithm in [4] for a big variety of examples, and our algorithm is particularly valuable for general cases when $k_i>1$ appears.

While, in this paper, we only deal with given planes in general position, the input data of planes for applications may not be so general. An approach common to practitioners of homotopies to solve a given problem is to deform the solutions of the general problem to those of the special problem by applying cheater's homotopy [5] or coefficient-parameter polynomial continuation [6, 7].

In [4], new homotopies were presented to compute p-plane producing curves intersecting m-planes at prescribed interpolation points. A future project would be to investigate whether the improvements proposed in this paper also apply to those new homotopies developed in [4].

2. Preliminaries.

DEFINITION 1. *Let* $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ *be a set of planes in* $\mathbb{C}^{m+p}$ *with* $\dim(A_i)=a_i$. *The set*

$$\Omega(A_1,\ldots,A_p) := \{\ p\text{-planes } X \text{ in } \mathbb{C}^{m+p} \,|\, \dim(X\bigcap A_i)\geq i, i=1,\ldots,p\ \}$$

is called a Schubert variety.

For planes $A_1 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq \cdots \subsetneq B_p$ with $\dim(B_i)=\dim(A_i)=a_i$, for $i=1,\ldots,p$, a nonsingular linear transformation in $\mathbb{C}^{m+p}$ can be constructed to transform A_i to B_i for $i=1,\ldots,p$, and the induced transformation transforms $\Omega(A_1,\ldots,A_p)$ onto $\Omega(B_1,\ldots,B_p)$. For this reason, the notation $\Omega(a_1,\ldots,a_p)$ is frequently used without specifying the planes A_i where $\dim(A_i)=a_i, i=1,\ldots,p$.

Now consider planes $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq \cdots \subsetneq B_p$ with $\dim(A_i)=a_i$ and $\dim(B_i)=b_i$, $i=1,\ldots,p$. When they are all in general position, we may assume

$$A_i=\langle e_1,\ldots,e_{a_i}\rangle \text{ and } B_i=\langle e_{m+p+1-b_i},\ldots,e_{m+p}\rangle, \quad i=1,\ldots,p,$$

where e_j is the unit vector in $\mathbb{C}^{m+p}$ with unit at the jth entry. Here, and from here on, $\langle v_1,\ldots,v_l\rangle$ denotes the plane spanned by $v_1,\ldots,v_l$. If $X\in\Omega(A_1,\ldots,A_p)$ $\bigcap\Omega(B_1,\ldots,B_p)$, then $\dim(X\bigcap A_{p+1-i})\geq p+1-i$ and $\dim(X\bigcap B_i)\geq i$. Thus, since $\dim(X)=p$ and both $X\bigcap A_{p+1-i}$ and $X\bigcap B_i$ are planes in X,

$$\begin{aligned}\dim(A_{p+1-i}\textstyle\bigcap B_i) &\geq \dim((X\textstyle\bigcap A_{p+1-i})\textstyle\bigcap(X\textstyle\bigcap B_i))\\ &\geq \dim(X\textstyle\bigcap A_{p+1-i})+\dim(X\textstyle\bigcap B_i)-\dim(X)\\ &\geq p+1-i+i-p=1.\end{aligned}$$

So, $a_{p+1-i} + b_i \geq m+p+1$. Conversely, if $a_{p+1-i} + b_i \geq m+p+1$, then $a_{p+1-i} \geq m+p+1-b_i$. Thus $e_{m+p+1-b_i} \in B_i \bigcap A_{p+1-i}$. Let $X = \langle e_{m+p+1-b_p}, e_{m+p+1-b_{p-1}}, \dots, e_{m+p+1-b_1} \rangle$. Obviously, $X \bigcap B_i = \langle e_{m+p+1-b_1}, e_{m+p+1-b_2}, \dots, e_{m+p+1-b_i} \rangle$ and $\dim(X \bigcap B_i) = i$. Furthermore, $X \bigcap A_{p+1-i} \supseteq \langle e_{m+p+1-b_p}, \dots, e_{m+p+1-b_i} \rangle$ and hence $\dim(X \bigcap A_{p+1-i}) \geq p+1-i$. Thus $X \in \Omega(A_1, \dots, A_p) \bigcap \Omega(B_1, \dots, B_p)$. Therefore, we have the following proposition.

PROPOSITION 1 (Theorem I, p. 327 [1]). *When $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq \cdots \subsetneq B_p$ are planes in general position in $\mathbb{C}^{m+p}$ with $\dim(A_i) = a_i$ and $\dim(B_i) = b_i$ for $i = 1, \dots, p$, then $\Omega(a_1, \dots, a_p)$ and $\Omega(b_1, \dots, b_p)$ intersect if and only if*

$$a_{p+1-i} + b_i \geq m+p+1 \qquad \textit{for} \quad i = 1, \dots, p.$$

As a corollary, we have the following proposition.

PROPOSITION 2 (Corollary, p. 328 [1]). *$\Omega(a_1, \dots, a_p) \bigcap \Omega(m+p+1-a_p, \dots, m+p+1-a_1)$ consists of a unique p-plane for given $1 \leq a_1 < \cdots < a_p \leq m+p$.*

EXAMPLE 1. *Let $m = p = 2$, $A_1 = \langle e_1, e_2 \rangle$, $A_2 = \langle e_1, e_2, e_3 \rangle$, $B_1 = \langle e_3, e_4 \rangle$, and $B_2 = \langle e_2, e_3, e_4 \rangle$. Then $\Omega(A_1, A_2) \bigcap \Omega(B_1, B_2) = \langle e_2, e_3 \rangle$.*

In the rest of the paper, when we write $\mathbf{a} = (a_1, \dots, a_p)$, those coordinates will satisfy $1 \leq a_1 < \cdots < a_p \leq m+p$. Because of the importance of Proposition 2, $\mathbf{a}^* = (m+p+1-a_p, \dots, m+p+1-a_1)$ is called the *dual* of $\mathbf{a} = (a_1, \cdots, a_p)$.

For $0 \leq h \leq m$, let $\sigma_h := \Omega(m+1-h, m+2, \dots, m+p)$, the set of p-planes that meet a given $(m+1-h)$-plane. Since every p-plane will meet any $(m+1)$-plane, σ_0 is the collection of all p-planes.

For $(a_1, \dots, a_p)$ and $(b_1, \dots, b_p)$ with $a_{p+1-i} \geq m+p+1-b_i$, for $i = 1, \dots, p$, let $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq \cdots \subsetneq B_p$ be planes in $\mathbb{C}^{m+p}$ with $\dim(A_i) = a_i$ and $\dim(B_i) = b_i$. If $X \in \Omega(a_1, \dots, a_p) \bigcap \Omega(b_1, \dots, b_p)$, then X meets $A_{p+1-i} \bigcap B_i$ for $i = 1, \dots, p$. Let D be the smallest plane containing $A_p \bigcap B_1, \dots, A_1 \bigcap B_p$. Then $X \subset D$ and

$$\begin{aligned} \dim(D) &\leq \dim(A_p \textstyle\bigcap B_1) + \cdots + \dim(A_1 \textstyle\bigcap B_p) \\ &= a_p + b_1 - (m+p) + \cdots + a + 1 + b_p - (m+p) \\ &= \textstyle\sum_{i=1}^{p} (a_i + b_i) - (m+p)p. \end{aligned}$$

Let $h = \sum a_i + \sum b_i - (m+p+1)p$. Clearly,

$$\dim(D) = h + p \Longleftrightarrow a_{p-i} < m+p+1-b_i \leq a_{p-i+1} \quad \forall i = 1, \dots, p.$$

When $\dim(D) = h+p$, let G_h be a generic $(m+1-h)$-plane. Representing D and G_h by matrices consisting of independent vectors in $\mathbb{C}^{m+p}$, the rank of the $(m+p) \times (m+p+1)$ matrix $[D|G_h]$ is $m+p$. Thus, up to a scalar factor, there is a unique nonzero vector $g \in G_h$, where $g = v_1 + \cdots + v_p$ with $v_i \in A_{p+1-i} \bigcap B_i$ for $i = 1, \dots, p$. Let $X = \langle v_1, \dots, v_p \rangle$; then $X \in \Omega(A_1, \dots, A_p) \bigcap \Omega(B_1, \dots, B_p)$ and meets G_h.

PROPOSITION 3 (Theorem III, p. 333 [1]). *Let $1 \leq h \leq m$. For $(a_1, \dots, a_p)$ and $(b_1, \dots, b_p)$ satisfying*

$$a_{p-i} < m+p+1-b_i \leq a_{p+1-i}, \quad h = \sum a_i + \sum b_i - (m+p+1)p, \tag{3}$$

the intersection $\Omega(a_1, \dots, a_p) \bigcap \Omega(b_1, \dots, b_p) \bigcap \sigma_h$ consists of a unique p-plane.

EXAMPLE 2. *Let $m = p = 2$, $A_1 = \langle e_1, e_2 \rangle$, $A_2 = \langle e_1, e_2, e_3 \rangle$, $B_1 = \langle e_3, e_4 \rangle$, $B_2 = \langle e_1, e_2, e_3, e_4 \rangle$, and $\sigma_1 = \Omega(D_1, D_2)$, where D_1 is a generic 2-plane and $D_2 = \mathbb{C}^4$. Then $A_1 \bigcap B_2 = A_1$ and $A_2 \bigcap B_1 = \langle e_3 \rangle$. Denote the 1-plane $D_1 \bigcap \langle A_1, e_3 \rangle$ by D_1'.*

Let $u = (u_1, u_2, u_3, u_4) \in D_1'$ *and* $f_1 = (u_1, u_2, 0, 0)$. *Then* $\Omega(2,3) \bigcap \Omega(2,4) \bigcap \sigma_1 = \langle f_1, e_3 \rangle$.

For $\Omega(a_1, \ldots, a_p)$ and $\Omega(b_1, \ldots, b_p)$, $\Omega(a_1, \ldots, a_p) + \Omega(b_1, \ldots, b_p)$ denotes the class of p-planes X, where for planes $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq \cdots \subsetneq B_p$ with $\dim(A_i) = a_i$ and $\dim(B_i) = b_i$ for $i = 1, \ldots, p$, $\dim(X \bigcap A_i) \geq i$ (or $\dim(X \bigcap B_i) \geq i$) for all $i = 1, \ldots, p$. We abbreviate $\Omega(a_1, \ldots, a_p) + \Omega(a_1, \ldots, a_p)$ by $2\Omega(a_1, \ldots, a_p)$ and in general

$$\sum_{i=1}^{d} \Omega(a_1, \ldots, a_p) := d\,\Omega(a_1, \ldots, a_p).$$

Furthermore, $\Omega(a_1, \ldots, a_p) \bullet \Omega(b_1, \ldots, b_p)$ represents the class of p-planes X, where $\dim(X \bigcap A_i) \geq i$ *and* $\dim(X \bigcap B_i) \geq i$ for all $i = 1, \ldots, p$.

For sets of p-planes A and B, we write $A \bullet B$ for $A \bigcap B$. We say A is *equivalent* to B, denoted by $A \sim B$, if whenever

$$A \bullet \Omega(\mathbf{c}) = k\,\Omega(1, \ldots, p)$$

for some $\mathbf{c} = (c_1, \ldots, c_p)$, we also have

$$B \bullet \Omega(\mathbf{c}) = k\,\Omega(1, \ldots, p).$$

Note that $\Omega(1, \ldots, p)$ represents a general p-plane. The following property [1, 2, 3] will be used repeatedly for the establishment of our algorithm:

$$A \sim B \Longrightarrow A \bullet \sigma_h \sim B \bullet \sigma_h.$$

Following Proposition 3, for fixed $a_1, \ldots, a_p$ and h, any $\mathbf{b} = (b_1, \ldots, b_p)$ satisfying (3) yields

$$\Omega(a_1, \ldots, a_p) \bullet \sigma_h \bullet \Omega(b_1, \ldots, b_p) = \Omega(1, \ldots, p).$$

On the other hand, for the dual $\mathbf{b}^* = (b_1^*, \ldots, b_p^*) = (m+p+1-b_p, \ldots, m+p+1-b_1)$ of $\mathbf{b}$, by Proposition 2,

$$\Omega(b_1^*, \ldots, b_p^*) \bullet \Omega(b_1, \ldots, b_p) = \Omega(1, \ldots, p).$$

Moreover, for $\bar{\mathbf{b}} = (\bar{b}_1, \ldots, \bar{b}_p)$ satisfying (3), but $\bar{\mathbf{b}} \neq \mathbf{b}^*$,

$$\Omega(\bar{b}_1, \ldots, \bar{b}_p) \bullet \Omega(b_1, \ldots, b_p) = \emptyset.$$

These observations lead to the following important formula.

PROPOSITION 4 (Pieri's formula, p. 354 [1]).

$$\Omega(a_1, \ldots, a_p) \bullet \sigma_h \sim \sum_{\mathbf{b}=(b_1,\ldots,b_p)} \Omega(b_1, \ldots, b_p), \textit{ where}$$

$$(4) \qquad 0 < b_1 \leq a_1 < b_2 \leq a_2 < \cdots \leq a_{p-1} < b_p \leq a_p \textit{ with } \sum b_j = \sum a_j - h.$$

When we fix $\mathbf{a} = (a_1, \ldots, a_p)$ and h, those $\mathbf{b} = (b_1, \ldots, b_p)$ satisfying (4) together with $\mathbf{a}$ are called the *Pieri nodes*; the nodes $\mathbf{b}$ are *induced* Pieri nodes of node $\mathbf{a}$.

From here on, we will use $[a_1, \ldots, a_p]$ to denote a Pieri node or its dual. Recall that for $i = 1, \ldots, n$, p-planes that meet plane L_i with $\dim(L_i) = m + 1 - k_i$ belong to $\Omega(m+1-k_i, m+2, \ldots, m+p) = \sigma_{k_i}$, and the condition $k_1 + \cdots + k_n = mp$ warrants

$$\sigma_{k_1} \bullet \sigma_{k_2} \bullet \cdots \bullet \sigma_{k_n} = d\,\Omega(1, \ldots, p). \tag{5}$$

Thus problem ($\star$) introduced in section 1 can now be interpreted as follows: finding all d specific p-planes in $\sigma_{k_1} \bullet \sigma_{k_2} \bullet \cdots \bullet \sigma_{k_n}$ for given planes $L_1, \ldots, L_n$ in general position. To calculate $\sigma_{k_1} \bullet \sigma_{k_2} \bullet \cdots \bullet \sigma_{k_n}$ in (5), Pieri's formula in Proposition 4 will be used as a main tool. The Pieri nodes derived in the process constitute a *Pieri poset*, and the number d is called the *Pieri root count.*

EXAMPLE 3. *For $m=2$, $p=2$ and given planes L_1, L_2, L_3, L_4 in general position with $\dim(L_i) = 2$ and $k_i = m + 1 - d_i = 1$ for all $i = 1, \ldots, 4$,*

$$\begin{aligned} & \sigma_{k_1} \bullet \sigma_{k_2} \bullet \sigma_{k_3} \bullet \sigma_{k_4} \\ = \;& \Omega(2,4) \bullet \sigma_{k_2} \bullet \sigma_{k_3} \bullet \sigma_{k_4} \\ \sim \;& (\Omega(1,4) + \Omega(2,3)) \bullet \sigma_{k_3} \bullet \sigma_{k_4} \\ \sim \;& 2\,\Omega(1,3) \bullet \sigma_{k_4} \\ \sim \;& 2\,\Omega(1,2). \end{aligned}$$

The Pieri poset of all the Pieri nodes and the poset that consists of their duals are shown in Figure 1.

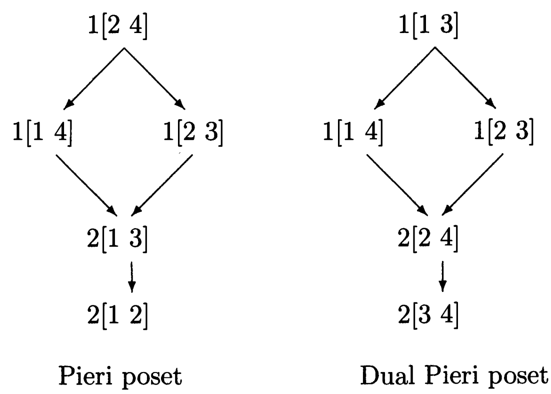

FIG. 1.

Now, any 2-plane X that meets L_1 must be in $\Omega(3,4) \bullet \sigma_1 \sim \Omega(2,4)$. Since $[2,4]^* = [1,3]$, there is a unique 2-plane in $\Omega(2,4) \bullet \Omega(1,3)$. So, if we let $A_1 = \langle e_1 \rangle$ and $A_2 = \langle e_1, e_2, e_3 \rangle$, there is a unique 2-plane in $\Omega(A_1, A_2)$ consisting of 2-planes of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & u \\ 0 & 0 \end{bmatrix} := X_{[1,3]}$$

that meet L_1. We may determine this unique $X_{[1,3]}$ by finding u via its intersection condition with L_1.

Similarly, any 2-plane X that meets both L_1 and L_2 must lie in $\Omega(3,4) \bullet \sigma_1 \bullet \sigma_1 \sim \Omega(1,4) + \Omega(2,3)$ by Proposition 4. Since $[1,4]^* = [1,4]$ and $\Omega(2,3) \bullet \Omega(1,4) = \emptyset$, by *letting* $A_1 = \langle e_1 \rangle$ *and* $A_2 = \langle e_1, e_2, e_3, e_4 \rangle$, *there is a unique* 2*-plane in* $\Omega(A_1, A_2)$ *consisting of* 2*-planes of the form*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & v_1 \\ 0 & v_2 \end{bmatrix} := X_{[1,4]}$$

that meet both L_1 *and* L_2. *We may determine this unique* $X_{[1,4]}$ *by finding* v_1 *and* v_2 *via the intersection conditions of meeting* L_1 *and* L_2. *On the other hand, since* $[2,3]^* = [2,3]$ *and* $\Omega(1,4) \bullet \Omega(2,3) = \emptyset$, *there is a unique* 2*-plane in* $\Omega(A_1, A_2)$ *with* $A_1 = \langle e_1, e_2 \rangle$ *and* $A_2 = \langle e_1, e_2, e_3 \rangle$ *consisting of* 2*-planes of the form*

$$\begin{bmatrix} 1 & 0 \\ v_1' & 1 \\ 0 & v_2' \\ 0 & 0 \end{bmatrix} := X_{[2,3]}$$

that meet L_1 *and* L_2. *This* $X_{[2,3]}$ *is decided when* v_1' *and* v_2' *are found.*

Continuing the same pattern, since $\Omega(3,4) \bullet \sigma_1 \bullet \sigma_1 \bullet \sigma_1 \sim 2\Omega(1,3)$ *and* $[1,3]^* = [2,4]$, *there are two* 2*-planes in* $\Omega(A_1, A_2)$, *with* $A_1 = \langle e_1, e_2 \rangle$ *and* $A_2 = \langle e_1, e_2, e_3, e_4 \rangle$, *consisting of* 2*-planes of the form*

$$\begin{bmatrix} 1 & 0 \\ w_1 & 1 \\ 0 & w_2 \\ 0 & w_3 \end{bmatrix} := X_{[2,4]}$$

that meet L_1, L_2, *and* L_3. *And,* $\Omega(3,4) \bullet \sigma_1 \bullet \sigma_1 \bullet \sigma_1 \bullet \sigma_1 \sim 2\Omega(1,2)$ *as well as* $[1,2]^* = [3,4]$ *imply that the two* 2*-planes that meet all* $L_1, \ldots, L_4$ *can be found by solving two set of* y*'s of*

$$\begin{bmatrix} 1 & 0 \\ y_1 & 1 \\ y_3 & y_2 \\ 0 & y_4 \end{bmatrix} := X_{[3,4]}$$

in $\Omega(A_1, A_2)$ *with* $A_1 = \langle e_1, e_2, e_3 \rangle$ *and* $\langle e_1, e_2, e_3, e_4 \rangle$.

The theme of the so-called Pieri homotopy algorithm is as follows:

1. *Finding* u *in* $X_{[1,3]}$ *by the criteria of meeting* L_1.
2. (a) *Solving* $\{v_1, v_2\}$ *in* $X_{[1,4]}$ *by a homotopy with a starting point containing* $\{v_1 = u, v_2 = 0\}$.
   (b) *Solving* $\{v_1, v_2\}$ *in* $X_{[2,3]}$ *by a different homotopy with a starting point containing* $\{v_1' = 0, v_2' = u\}$.
3. *Solving two sets of* $\{w_1, w_2, w_3\}$ *in* $X_{[2,4]}$ *by a homotopy with two starting points containing* $\{w_1 = 0, w_2 = v_1, w_3 = v_2\}$ *and* $\{w_1 = v_1', w_2 = v_2' w_3 = 0\}$, *respectively.*
4. *Solving two sets of* $\{y_1, y_2, y_3, y_4\}$ *in* $X_{[3,4]}$ *by a homotopy with two starting points containing* $\{y_1 = w_1, y_2 = w_2, y_3 = w_3, y_4 = 0\}$ *with two sets of values of* $\{w_1, w_2, w_3\}$ *obtained at the last step.*

The details of those homotopies of our approach will be elaborated in the next section.

From the process in the above example, we solve the ultimate solutions in $X_{[3,4]}$ by following the cascade of solving

$$
\begin{tikzcd}
 & X_{[1\,3]} \arrow[dl] \arrow[dr] & \\
X_{[2\,3]} \arrow[dr] & & X_{[1\,4]} \arrow[dl] \\
 & 2X_{[2\,4]} \arrow[d] & \\
 & 2X_{[3\,4]}. &
\end{tikzcd}
\tag{6}
$$

For $\mathbf{a} = [a_1, \ldots, a_p]$, write

$$
X_{\mathbf{a}} = X_{[a_1,\ldots,a_p]} := \left[\begin{array}{ccc}
1 & & 0 \\
x_{1,1} & \ddots & \\
\vdots & \ddots & 1 \\
x_{(a_1-1),1} & & x_{1,p} \\
 & \ddots & \vdots \\
 & & x_{(a_p-p),p} \\
 & & 0 \\
0 & & \vdots \\
 & & 0
\end{array}\right].
$$

Those $\mathbf{a}$'s in (6) actually follow the duals of the Pieri poset in Figure 1. For nodes $\mathbf{a}$ and $\mathbf{b}$,

$$\mathbf{a} \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \mathbf{b}$$

is called a *chain* joining $\mathbf{a}$ and $\mathbf{b}$. A chain joining $\mathbf{a} = [m+1, m+2, \ldots, m+p]$ and $\mathbf{b} = [1, \ldots, p]$ is called a *complete chain.* The Pieri homotopy algorithms in general are constructed based on the duals of the Pieri poset consisting of all the derived Pieri nodes.

3. Algorithms. For given planes $L_1, \ldots, L_n$ in $\mathbb{C}^{m+p}$ in general position with $\dim(L_i) = m + 1 - k_i$ for $i = 1, \ldots, n$, all derived Pieri nodes in

$$\sigma_{k_1} \bullet \sigma_{k_2} \bullet \cdots \bullet \sigma_{k_n}$$

form a poset. Unless otherwise indicated, we shall use the term "Pieri poset" for the poset of duals of all those Pieri nodes. As mentioned in the introduction, we shall represent each L_i by a $(p + k_i - 1) \times (m + p)$ matrix K_i whose rows consist of all the normals of the linear equations that define L_i.

(a) Hypersurface intersection conditions, where $k_i = 1$ for all $i = 1, \dots, n$. Letting $\mathbf{a}^0 = [1, 2, \dots, p]$ sit on top of the Pieri poset, we may write

$$\mathbf{a}^0 \to \mathbf{a}^1 \to \cdots \to \mathbf{a}^n \tag{7}$$

for a complete chain in the poset, and it is obvious that the coordinates of consecutive nodes $\mathbf{a}^j$ and $\mathbf{a}^{j+1}$ in the chain can differ by 1 on only one component. We shall use

$$\mathbf{a}^j \xrightarrow{\mu_{j+1}} \mathbf{a}^{j+1}$$

to denote that the μ_{j+1}th component of $\mathbf{a}^j$ is increased by 1 to reach $\mathbf{a}^{j+1}$. We may therefore write

$$\mathbf{a}^0 \xrightarrow{\mu_1} \mathbf{a}^1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_n} \mathbf{a}^n$$

for a complete chain. Recall that for $\mathbf{a} = [a_1, \dots, a_p]$

$$X_{\mathbf{a}} = \begin{bmatrix} 1 & & 0 \\ x_{1,1} & \ddots & \\ \vdots & \ddots & 1 \\ x_{(a_1-1),1} & & x_{1,p} \\ & \ddots & \vdots \\ & & x_{(a_p-p),p} \\ & & 0 \\ 0 & & \vdots \\ & & 0 \end{bmatrix}.$$

For $\mathbf{a}^0 \xrightarrow{\mu_1} \mathbf{a}^1$, the only unknown in $X_{\mathbf{a}^1}$ can be determined by

$$K_1 X_{\mathbf{a}^1} \Lambda_1^1 = 0,$$

where $\Lambda_1^1 = e_{\mu_1} \in \mathbb{C}^p$. Now, suppose we have proceeded up to

$$\mathbf{a}^0 \xrightarrow{\mu_1} \mathbf{a}^1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_j} \mathbf{a}^j$$

in the chain. This means that we have solved all the variables in $X_{\mathbf{a}^j}$ and found $\Lambda_1^j, \dots, \Lambda_j^j \in \mathbb{P}^{p-1}$ such that

$$K_l X_{\mathbf{a}^j} \Lambda_l^j = 0 \quad \text{for} \quad l = 1, \dots, j.$$

Namely, a p-plane in the form $X_{\mathbf{a}^j}$ that meets planes $L_1, \dots, L_j$ has been determined. To proceed one step further in the chain, for

$$\mathbf{a}^j \xrightarrow{\mu_{j+1}} \mathbf{a}^{j+1}, \quad \text{where } \mathbf{a}^{j+1} = [a_1^{(j+1)}, \dots, a_p^{(j+1)}],$$

consider the homotopy

$$H(t, X_{\mathbf{a}^{j+1}}, \Lambda^{j+1}) = \begin{cases} K_1 X_{\mathbf{a}^{j+1}} \Lambda_1^{j+1} & = 0, \\ \vdots & \\ K_j X_{\mathbf{a}^{j+1}} \Lambda_j^{j+1} & = 0, \\ [(1-t)\hat{K}_{\mathbf{a}^{j+1}} + tK_{j+1}] X_{\mathbf{a}^{j+1}} \Lambda_{j+1}^{j+1} & = 0, \end{cases} \tag{8}$$

where the μ_lth component of Λ_l^{j+1} is 1 for $l = 1, \ldots, j+1$, and $\widehat{K}_{\mathbf{a}^{j+1}}$ is the matrix $[e_{a_1^{(j+1)}}, \ldots, e_{a_p^{(j+1)}}]^T$. For each $t \in [0,1]$, the system admits $p-1$ variables in Λ_l^{j+1} for each $l = 1, \ldots, j+1$ and $j+1$ variables in $X_{\mathbf{a}^{j+1}}$; it admits, in total, $(p-1)(j+1) + (j+1) = p(j+1)$ variables. It is clear that the total number of equations is also $p(j+1)$, making the system a square system. When $t = 0$,

$$\begin{aligned} X_{\mathbf{a}^{j+1}} &= X_{\mathbf{a}^j}, \\ \Lambda_l^{j+1} &= \Lambda_l^j, \qquad l = 1, \ldots, j, \\ \Lambda_{j+1}^{j+1} &= e_{\mu_{j+1}} (\in \mathbb{C}^p) \end{aligned}$$

is a solution of the system $H(0, X_{\mathbf{a}^{j+1}}, \Lambda^{j+1}) = 0$ in (8). Following the homotopy path of $H(t, X_{\mathbf{a}^{j+1}}, \Lambda^{j+1}) = 0$ emanating from this solution, we obtain a solution of $X_{\mathbf{a}^{j+1}}$ and Λ_l^{j+1} for $l = 1, \ldots, j+1$ at $t = 1$ that satisfies

$$K_l X_{\mathbf{a}^{j+1}} \Lambda_l^{j+1} = 0 \quad \text{for } l = 1, \ldots, j+1.$$

A p-plane that meets $L_1, \ldots, L_{j+1}$ in the form of $X_{\mathbf{a}^{j+1}}$ is then found and the chain has been extended one step further; namely, we have proceeded along the chain up to

$$\mathbf{a}^0 \xrightarrow{\mu_1} \mathbf{a}^1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{j+1}} \mathbf{a}^{j+1}.$$

When we proceed further along the chain and arrive at $\mathbf{a}^n$, a p-plane that meets all L_i, $i = 1, \ldots, n$, becomes available.

EXAMPLE 4. *In Example 3, there are two chains in the dual poset:*

$$\textit{chain 1: } [1\,2] \xrightarrow{2} [1\,3] \xrightarrow{2} [1\,4] \xrightarrow{1} [2\,4] \xrightarrow{1} [3\,4],$$
$$\textit{chain 2: } [1\,2] \xrightarrow{2} [1\,3] \xrightarrow{1} [2\,3] \xrightarrow{2} [2\,4] \xrightarrow{1} [3\,4],$$

and the corresponding homotopies are

$$\begin{array}{ll}
[1\,2] & [1\,2] \\
\downarrow & \downarrow \\
[1\,3]: \left[\; K_1 X_{[1\,3]} \Lambda_1^1 = 0, \Lambda_1^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] & [1\,3]: \left[\; K_1 X_{[1\,3]} \Lambda_1^1 = 0, \Lambda_1^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \\
\downarrow & \downarrow \\
[1\,4]: \left[\begin{array}{r} K_1 X_{[1\,4]} \Lambda_1^2 = 0 \\ \{(1-t)\underline{[e_1, e_4]^T} + tK_2\} X_{[1\,4]} \Lambda_2^2 = 0 \end{array} \right. & [2\,3]: \left[\begin{array}{r} K_1 X_{[2\,3]} \Lambda_1^2 = 0 \\ \{(1-t)\underline{[e_2, e_3]^T} + tK_2\} X_{[2\,3]} \Lambda_2^2 = 0 \end{array} \right. \\
\downarrow & \downarrow \\
[2\,4]: \left[\begin{array}{r} K_1 X_{[2\,4]} \Lambda_1^3 = 0 \\ K_2 X_{[2\,4]} \Lambda_2^3 = 0 \\ \{(1-t)[e_2, e_4]^T + tK_3\} X_{[2\,4]} \Lambda_3^3 = 0 \end{array} \right. & [2\,4]: \left[\begin{array}{r} K_1 X_{[2\,4]} \Lambda_1^3 = 0 \\ K_2 X_{[2\,4]} \Lambda_2^3 = 0 \\ \{(1-t)[e_2, e_4]^T + tK_3\} X_{[2\,4]} \Lambda_3^3 = 0 \end{array} \right. \\
\downarrow & \downarrow \\
[3\,4]: \left[\begin{array}{r} K_1 X_{[3\,4]} \Lambda_1^4 = 0 \\ K_2 X_{[3\,4]} \Lambda_2^4 = 0 \\ K_3 X_{[3\,4]} \Lambda_3^4 = 0 \\ \{(1-t)[e_3, e_4]^T + tK_4\} X_{[3\,4]} \Lambda_4^4 = 0 \end{array} \right. & [3\,4]: \left[\begin{array}{r} K_1 X_{[3\,4]} \Lambda_1^4 = 0 \\ K_2 X_{[3\,4]} \Lambda_2^4 = 0 \\ K_3 X_{[3\,4]} \Lambda_3^4 = 0 \\ \{(1-t)[e_3, e_4]^T + tK_4\} X_{[3\,4]} \Lambda_4^4 = 0. \end{array} \right.
\end{array}$$

For chain 1, $\Lambda_k^l = \left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$, $k = 1, 2$, $l = 1, \ldots, k$, $\Lambda_k^l = \left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$, $k = 3, 4$, $l = 1, \ldots, k$; *for chain* 2, $\Lambda_k^l = \left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$, $k = 1, 3$, $l = 1, \ldots, k$, $\Lambda_k^l = \left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$, $k = 2, 4$, $l = 1, \ldots, k$. *Between chain* 1 *and* 2, *the only distinct homotopies are* $[1\,3] \xrightarrow{2} [1\,4]$ *in chain* 1 *and* $[1\,3] \xrightarrow{1} [2\,3]$ *in chain* 2. □

EXAMPLE 5. *For* $m = 3$, $p = 2$, *let* $L_1, \ldots, L_6$ *be planes with* $\dim(L_i) = 3$ *for* $i = 1, \ldots, 6$. *The Pieri poset is shown in Figure* 2. *For the complete chain*

$$[1\,2] \xrightarrow{2} [1\,3] \xrightarrow{2} [1\,4] \xrightarrow{1} [2\,4] \xrightarrow{2} [2\,5] \xrightarrow{1} [3\,5] \xrightarrow{1} [4\,5],$$

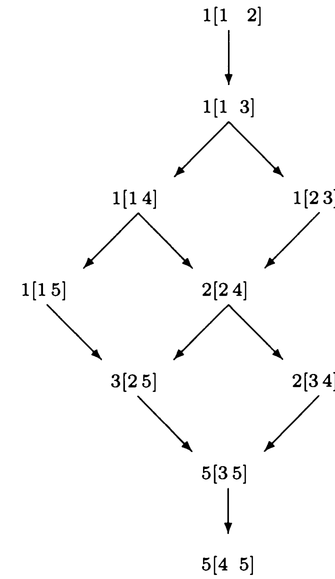

FIG. 2.

the homotopies are

$$
\begin{array}{ll}
[1\,2] & \\
\downarrow & \\
[1\,3] & : \left[\; K_1 X_{[1\,3]} \Lambda_1^1, = 0, \Lambda_1^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right. \\
\downarrow & \\
[1\,4] & : \left[\begin{array}{r} K_1 X_{[1\,4]} \Lambda_1^2 = 0, \\ \left\{(1-t)[e_1, e_4]^T + tK_2\right\} X_{[1\,4]} \Lambda_2^2 = 0, \end{array} \right. \\
\downarrow & \\
[2\,4] & : \left[\begin{array}{rl} K_l X_{[2\,4]} \Lambda_l^3 = 0, & l = 1, 2, \\ \left\{(1-t)[e_2, e_4]^T + tK_3\right\} X_{[2\,4]} \Lambda_3^3 = 0, & \end{array} \right. \\
\downarrow & \\
[2\,5] & : \left[\begin{array}{rl} K_l X_{[2\,5]} \Lambda_l^4 = 0, & l = 1, 2, 3, \\ \left\{(1-t)[e_2, e_5]^T + tK_4\right\} X_{[2\,5]} \Lambda_4^4 = 0, & \end{array} \right. \\
\downarrow & \\
[3\,5] & : \left[\begin{array}{rl} K_l X_{[3\,5]} \Lambda_l^5 = 0, & l = 1, 2, 3, 4, \\ \left\{(1-t)[e_3, e_5]^T + tK_5\right\} X_{[3\,5]} \Lambda_5^5 = 0, & \end{array} \right. \\
\downarrow & \\
[4\,5] & : \left[\begin{array}{rl} K_l X_{[4\,5]} \Lambda_l^6 = 0, & l = 1, 2, 3, 4, 5, \\ \left\{(1-t)[e_4, e_5]^T + tK_6\right\} X_{[4\,5]} \Lambda_6^6 = 0, & \end{array} \right.
\end{array}
$$

where $\Lambda_1^l = \left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$, $\Lambda_2^l = \left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$, $\Lambda_3^l = \left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$, $\Lambda_4^l = \left[\begin{smallmatrix} * \\ 1 \end{smallmatrix}\right]$, $\Lambda_5^l = \left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$, *and* $\Lambda_6^l = \left[\begin{smallmatrix} 1 \\ * \end{smallmatrix}\right]$. □

Remark 1. Let $\mathbf{a}$ be a node shared by k different complete chains

$$\mathbf{a}_l^0 \xrightarrow{\mu_{1l}} \mathbf{a}_l^1 \xrightarrow{\mu_{2l}} \cdots \xrightarrow{\mu_{nl}} \mathbf{a}_l^n, \quad l = 1, \ldots, k.$$

Say $\mathbf{a}_l^{j+1} = \mathbf{a}$, for $l = 1, \ldots, k$. This means $\sigma_{k_1} \bullet \sigma_{k_2} \bullet \cdots \bullet \sigma_{k_j} \bullet \Omega(\mathbf{a}) = k\,\Omega(1, \ldots, p)$, where $k_i = 1$ for $i = 1, \ldots, j$. In this situation, the homotopies for the extensions $\mathbf{a}_l^j \xrightarrow{\mu_{(j+1)l}} \mathbf{a}_l^{j+1} = \mathbf{a}$ in (8) are the same for all l. It is critically important that those

k paths that emanate from k different starting points

$$X_{\mathbf{a}} = X_{\mathbf{a}_l^j}, \quad \Lambda_{il}^{j+1} = \Lambda_{il}^{j} \text{ for } i = 1, \dots, j, \quad \Lambda_{(j+1)l}^{j+1} = \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \mu_{(j+1)l}\text{th} \\ \\ \\ \end{array} \end{array}$$

will reach different solutions at $t = 1$. This assertion is warranted by the following observations. Since for each $t \in [0,1]$, $(1-t)\widehat{K}_{\mathbf{a}} + tK_{j+1}$ represents an m-plane $L_{j+1}(t)$, and those m-planes $L_{j+1}(t)$ are in general position for $0 < t \leq 1$, it follows that for each $t \in (0,1]$ the system

$$\begin{aligned} K_1 X_{\mathbf{a}} \Lambda_1^{j+1} &= 0, \\ &\ \vdots \\ K_j X_{\mathbf{a}} \Lambda_j^{j+1} &= 0, \\ [(1-t)\widehat{K}_{\mathbf{a}} + tK_{j+1}] X_{\mathbf{a}} \Lambda_{j+1}^{j+1} &= 0 \end{aligned}$$

has k solutions and all of them are nonsingular. Since at $t = 0$ those k solutions are also nonsingular, those k different paths of the same homotopy will lead to k different solutions at $t = 1$.

(b) General intersection conditions, where $k_i > 1$ for certain $1 \leq i \leq n$. The Pieri poset in this case is somewhat more complicated. For $k_i > 1$, let $\mathbf{a}^i$ be a derived node of $\mathbf{a}^{i-1}$. The coordinates of nodes $\mathbf{a}^{i-1}$ and $\mathbf{a}^i$ may have several different components and their differences may not simply differ by just 1. Moreover, as the following example shows, not all the nodes can be proceeded to reach final node $\mathbf{a}^n$ to be part of a complete chain.

EXAMPLE 6. *For $m = 5$, $p = 3$, and given planes $L_1, \dots, L_5$ in general position with* $\dim(L_i) = 3$, *for all $i = 1, \dots, 5$, $\sum_{i=1}^5 k_i = 15 = mp$. Furthermore,*

$$\begin{aligned} & \sigma_0 \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \\ \sim\ & [\Omega(6,7,8) \bullet \sigma_3] \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \\ \sim\ & [\Omega(3,7,8) \bullet \sigma_3] \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \\ \sim\ & [\Omega(1,6,8) + \Omega(2,5,8) + \Omega(3,4,8)] \bullet \sigma_3 \bullet \sigma_3 \bullet \sigma_3 \\ \sim\ & [2\Omega(1,3,8) + 3\Omega(1,4,7) + 2\Omega(2,4,6) + \Omega(1,5,6) + \Omega(3,4,5)] \bullet \sigma_3 \bullet \sigma_3 \\ \sim\ & [7\Omega(1,3,5) + 6\Omega(1,2,6)] \bullet \sigma_3 \\ \sim\ & 6\Omega(1,2,3). \end{aligned}$$

The Pieri poset in this case is shown in Figure 3, *and the poset consisting of complete chains is shown in Figure* 4.

Of course, only complete chains in the Pieri poset are meaningful in computing our solutions. For $\mathbf{a}^0 = (1, \dots, p)$, let

$$\mathbf{a}^0 \longrightarrow \mathbf{a}^1 \longrightarrow \cdots \longrightarrow \mathbf{a}^n$$

be a complete chain, where $\mathbf{a}^{j+1}$ is derived from $\mathbf{a}^j$ via $\sigma_{k_{j+1}}$ for $j = 1, \dots, n-1$. Namely, $\Omega(\mathbf{a}^j) \subset \sigma_0 \bullet \sigma_{k_1} \bullet \cdots \sigma_{k_j}$ and $\Omega(\mathbf{a}^{j+1}) \subset \sigma_0 \bullet \sigma_{k_1} \bullet \cdots \sigma_{k_{j+1}}$. When $k_i > 1$ for certain $i \in \{1, \dots, n\}$, we will insert artificial *intermediate* nodes between nodes $\mathbf{a}^{i-1}$ and $\mathbf{a}^i$ for our algorithm as follows:

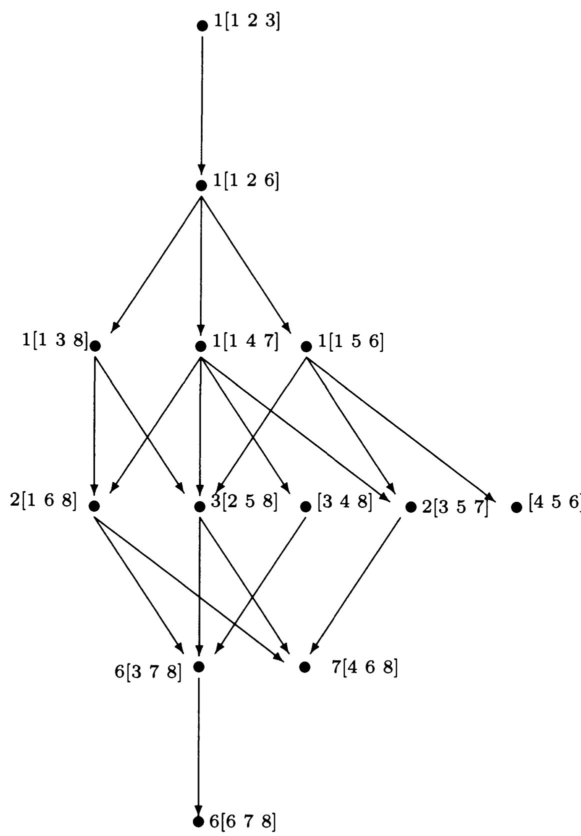

FIG. 3.

Writing

$$\mathbf{a}^{i-1} = [a_1^{(i-1)}, \dots, a_p^{(i-1)}] \quad \text{and} \quad \mathbf{a}^i = [a_1^{(i)}, \dots, a_p^{(i)}],$$

we let $l_1 = \min\{j \mid a_j^{(i-1)} < a_j^{(i)}\}$ and $\mathbf{b}^1 = (b_1^{(1)}, \dots, b_p^{(1)})$, where

$$b_j^{(1)} = \begin{cases} a_j^{(i-1)} & \text{for } j = 1, \dots, l_1 - 1, \\ a_{l_1}^{(i-1)} + 1 & \text{for } j = l_1 \\ a_j^{(i)} & \text{for } j = l_1 + 1, \dots, p. \end{cases}$$

Inductively, when $\mathbf{b}^s = (b_1^{(s)}, \dots, b_p^{(s)})$ is defined for $s < k_i - 1$, let $l_{s+1} = \min\{j \mid b_j^{(s)} < a_j^{(i)}\}$ and $\mathbf{b}^{s+1} = (b_1^{(s+1)}, \dots, b_p^{(s+1)})$, where

$$b_j^{(s+1)} = \begin{cases} b_j^{(s)} & \text{for } j = 1, \dots, l_{s+1} - 1, \\ b_{l_{s+1}}^{(s)} + 1 & \text{for } j = l_{s+1} \\ a_j^{(i)} & \text{for } j = l_{s+1} + 1, \dots, p. \end{cases}$$

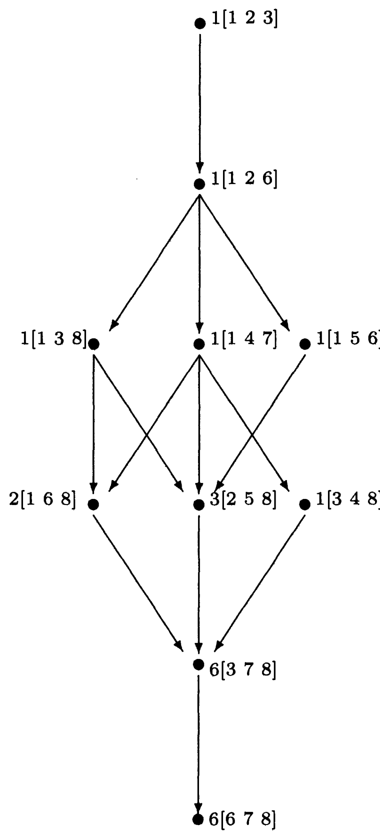

FIG. 4.

We insert those nodes $\mathbf{b}^1, \ldots, \mathbf{b}^{k_i-1}$ defined above between $\mathbf{a}^{i-1}$ and $\mathbf{a}^i$. Obviously, the coordinates of any two consecutive nodes among them can differ by 1 on only one coordinate. Therefore, we may write

$$\mathbf{a}^{i-1} := \mathbf{b}^0 \xrightarrow{\mu_1} \mathbf{b}^1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k_i-1}} \mathbf{b}^{k_i} := \mathbf{a}^i,$$

where μ_j in $\mathbf{b}^{j-1} \xrightarrow{\mu_j} \mathbf{b}^j$ represents the coordinate where $\mathbf{b}^{j-1}$ and $\mathbf{b}^j$ differ.

EXAMPLE 7. *For instance, the node insertion between* [1 4 7] *and* [1 6 8] *on Figure* 4 *of Example* 6 *is*

$$[1\ 4\ 7] \xrightarrow{2} (1\ 5\ 7) \xrightarrow{2} (1\ 6\ 7) \xrightarrow{3} [1\ 6\ 8],$$

and when all intermediate nodes are inserted the poset with complete chains is shown in Figure 5.

For consecutive nodes $\mathbf{a}^{i-1}$ and $\mathbf{a}^i$ with $k_i > 1$ and the chain joining the intermediate nodes between them,

$$\mathbf{a}^{i-1} = \mathbf{b}^0 \xrightarrow{\mu_1} \mathbf{b}^1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{k_i-1}} \mathbf{b}^{k_i} = \mathbf{a}^i, \tag{9}$$

suppose we have solved all the variables in $X_{\mathbf{a}^{i-1}}$ as well as $\Lambda_l^{i-1} \in \mathbb{P}^{p-1}$ for $l = 1, \ldots, i-1$ for which

$$K_l X_{\mathbf{a}^{i-1}} \Lambda_l^{i-1} = 0 \qquad \text{for} \quad l = 1, \ldots, i-1. \tag{10}$$

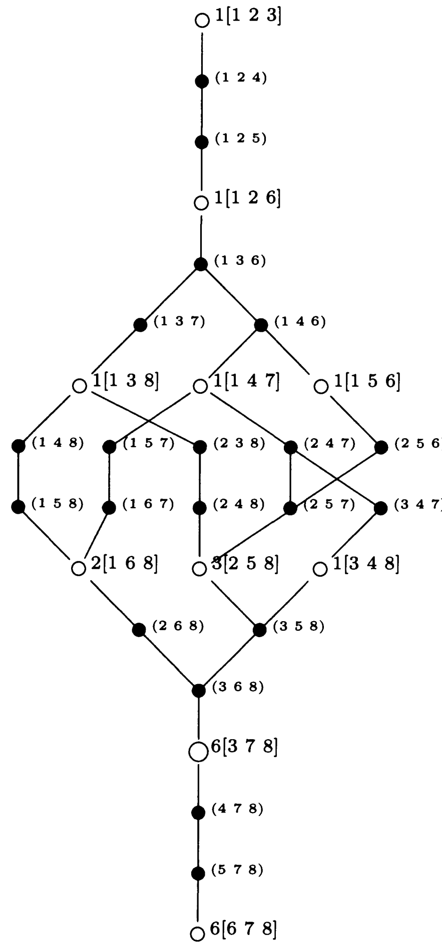

FIG. 5.

Recall that the $(p+k_j-1)\times(m+p)$ matrix K_l is a representation of the plane L_l. Let $K_i=[v_1,\dots,v_{p+k_i-1}]^T$, where v_s for $s=1,\dots,p+k_i-1$ are linearly independent vectors in $\mathbb{C}^{m+p}$. For

$$\mathbf{a}^{i-1}=\mathbf{b}^0 \xrightarrow{\mu_1} \mathbf{b}^1$$

consider the homotopy

$$(11)\qquad \begin{aligned} K_1X_{\mathbf{b}^1}\Lambda_1^{i+k_i-1} &= 0,\\ &\vdots\\ K_{i-1}X_{\mathbf{b}^1}\Lambda_{i-1}^{i+k_i-1} &= 0,\\ [(1-t)\hat{K}_1^0+t\hat{K}_1^1]X_{\mathbf{b}^1}\Lambda_{i+k_i-1}^{i+k_i-1} &= 0, \end{aligned}$$

where for $\mathbf{b}^1=(b_1^{(1)},\dots,b_p^{(1)})$

$$\begin{aligned} \hat{K}_1^0 &:= [e_{b_1^{(1)}},\dots,e_{b_p^{(1)}}]^T\\ \text{and}\quad \hat{K}_1^1 &:= [e_{b_1^{(1)}},\dots,e_{b_{\mu_1-1}^{(1)}},v_1,e_{b_{\mu_1+1}^{(1)}},\dots,e_{b_p^{(1)}}]^T. \end{aligned}$$

Moreover, the μ_1th coordinate of $\Lambda_{i+k_i-1}^{i+k_i-1} \in \mathbb{P}^{p-1}$ is set to be 1, and for $l = 1, \ldots, i-1$ the coordinate of $\Lambda_l^{i+k_i-1} \in \mathbb{P}^{p-1}$ is set to be 1 if the same coordinate of $\Lambda_l^{i-1} \in \mathbb{P}^{p-1}$ is 1.

This homotopy is a deformation of square systems of size

$$p + \sum_{l=1}^{i-1} (p + k_l - 1).$$

Clearly, when $t = 0$ any solution $X_{\mathbf{a}^{i-1}}$, $\Lambda_l^{i+k_i-1}$ for $l = 1, \ldots, i-1$ of (10) coupled with $\Lambda_{i+k_i-1}^{i+k_i-1} = e_{\mu_1}$ is a solution of (11). The solutions we obtain at $t = 1$ by following the paths of the homotopy in (11) emanating from those solutions will be established as solutions of the start system of the homotopy constructed for the next step.

Inductively, write $\mathbf{b}^l = (b_1^{(l)}, \ldots, b_p^{(l)})$ for $l = 0, \ldots, k_i$ and suppose for $2 \le j \le k_i$ the system

$$\begin{aligned} K_1 X_{\mathbf{b}^{j-1}} \Lambda_1^{i+k_i-(j-1)} &= 0, \\ &\vdots \\ K_{i-1} X_{\mathbf{b}^{j-1}} \Lambda_{i-1}^{i+k_i-(j-1)} &= 0, \\ \hat{K}_{j-1}^1 X_{\mathbf{b}^{j-1}} \Lambda_{i+k_i-(j-1)}^{i+k_i-(j-1)} &= 0, \end{aligned} \tag{12}$$

where

$$\hat{K}_{j-1}^1 := [e_{b_1^{(j-1)}}, \ldots, e_{b_{\mu_{j-1}-1}^{(j-1)}}, v_{j-1}, e_{b_{\mu_{j-1}+1}^{(j-1)}}, \ldots, e_{b_p^{(j-1)}}, v_1, \ldots, v_{j-2}]^T$$

has been solved. For

$$\mathbf{b}^{j-1} \xrightarrow{\mu_j} \mathbf{b}^j$$

consider the homotopy

$$\begin{aligned} K_1 X_{\mathbf{b}^j} \Lambda_1^{i+k_i-j} &= 0, \\ &\vdots \\ K_{i-1} X_{\mathbf{b}^j} \Lambda_{i-1}^{i+k_i-j} &= 0, \\ [(1-t)\hat{K}_j^0 + t\hat{K}_j^1] X_{\mathbf{b}^j} \Lambda_{i+k_i-j}^{i+k_i-j} &= 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \hat{K}_1^0 &= [e_{b_1^{(j)}}, \ldots, e_{b_p^{(j)}}, v_1, \ldots, v_{j-1}]^T \\ \text{and} \quad \hat{K}_j^1 &= [e_{b_1^{(j)}}, \ldots, e_{b_{\mu_j-1}^{(j)}}, v_j, e_{b_{\mu_j+1}^{(j)}}, \ldots, e_{b_p^{(j)}}, v_1, \ldots, v_{j-1}]^T. \end{aligned}$$

And, as in (11), the μ_jth coordinate of $\Lambda_{i+k_i-j}^{i+k_i-j} \in \mathbb{P}^{p-1}$ is set to be 1, and for $l = 1, \ldots, i-1$, the coordinate of $\Lambda_l^{i+k_i-j}$ is set to be 1 if the same coordinate of $\Lambda_l^{i+k_i-(j-1)}$ is 1.

This homotopy is a deformation of square system of size

$$p + j - 1 + \sum_{l=1}^{i-1} (p + k_l - 1),$$

and it is straightforward that any solution of the system in (12) induces a solution of (13) when $t = 0$. Those paths of the homotopy in (13) emanating from those solutions lead to, at $t = 1$, a set of solutions of

$$
\begin{aligned}
K_1 X_{\mathbf{b}^j} \Lambda_1^{i+k_i-j} &= 0,\\
&\vdots\\
K_{i-1} X_{\mathbf{b}^j} \Lambda_{i-1}^{i+k_i-j} &= 0,\\
\hat{K}_j^1 X_{\mathbf{b}^j} \Lambda_{i+k_i-j}^{i+k_i-j} &= 0.
\end{aligned} \tag{14}
$$

Continuing those steps successively from $j = 2$, when we reach $j = k_i$, the solutions at $t = 1$ provide a set of p-planes in the form $X_{\mathbf{a}^i}$ that meet $L_1, \ldots, L_i$.

EXAMPLE 8. *In Example 7, the homotopies of the chain*

$$
[1\,2\,3]\underbrace{\xrightarrow{3}(1\,2\,4)\xrightarrow{3}(1\,2\,5)\xrightarrow{3}[1\,2\,6]}_{\text{level 1}}
$$

$$
\underbrace{\xrightarrow{2}(1\,3\,6)\xrightarrow{2}(1\,4\,6)\xrightarrow{3}[1\,4\,7]}_{\text{level 2}}
$$

$$
\underbrace{\xrightarrow{1}(2\,4\,7)\xrightarrow{2}(2\,5\,7)\xrightarrow{3}[2\,5\,8]}_{\text{level 3}}\longrightarrow\cdots
$$

at the third level with $K_3 := [v_1, v_2, v_3, v_4, v_5]^T$ *are*

$$
\begin{array}{cl}
[1\,4\,7] & \\
\downarrow & \\
(2\,4\,7) & : \left[\begin{array}{r} K_1 X_{(2,4,7)}\Lambda_1^5 = 0,\\ K_2 X_{(2,4,7)}\Lambda_2^5 = 0,\\ \{(1-t)[e_2,e_4,e_7]^T + t[e_2,v_5,e_7]^T\}X_{(2,4,7)}\Lambda_5^5 = 0,\end{array}\right.\\
\downarrow & \\
(2\,5\,7) & : \left[\begin{array}{r} K_1 X_{(2,5,7)}\Lambda_1^4 = 0,\\ K_2 X_{(2,5,7)}\Lambda_2^4 = 0,\\ \{(1-t)[e_2,e_5,e_7,v_5]^T + t[e_2,e_5,v_4,v_5]^T\}X_{(2,5,7)}\Lambda_4^4 = 0,\end{array}\right.\\
\downarrow & \\
[2\,5\,8] & : \left[\begin{array}{r} K_1 X_{[2,5,8]}\Lambda_1^3 = 0,\\ K_2 X_{[2,5,8]}\Lambda_2^3 = 0,\\ \{(1-t)[e_2,e_5,e_8,v_4,v_5] + t[v_1,v_2,v_3,v_4,v_5]\}X_{[2,5,8]}\Lambda_3^3 = 0,\end{array}\right.
\end{array}
$$

where

$$
\Lambda_1^5, \Lambda_1^4, \Lambda_1^3 = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}, \quad \Lambda_2^5, \Lambda_2^4, \Lambda_2^3 = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix},
$$

$$
\textit{and} \quad \Lambda_5^5 = \begin{bmatrix} 1 \\ * \\ * \end{bmatrix}, \quad \Lambda_4^4 = \begin{bmatrix} * \\ 1 \\ * \end{bmatrix}, \quad \Lambda_3^3 = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}.
$$

Write $\widehat{K}_1^t = (1-t)[e_2, e_4, e_7]^T + t[e_2, v_5, e_7]^T$. *Then,*

$$\widehat{K}_1^t X_{(247)}\Lambda_5^5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ & x_2 & 1 \\ & x_3 & x_4 \\ & & x_5 \\ \mathbf{0} & & x_6 \\ & & x_7 \\ & & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 1 & 0 \\ \star & \star & \star \\ 0 & 0 & x_7 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Obviously, $\lambda_3 = 0$ *for all* $t \in [0,1]$, *and, as assigned,* $\lambda_1 = 1$ *for all* $t \in [0,1]$. *Let* $x_1^{(1)}, \ldots, x_7^{(1)}, \lambda_1^{(1)}(=1), \lambda_2^{(1)}, \lambda_3^{(1)}(=0)$ *be a solution of* $\widehat{K}_1^1 X_{(247)}\Lambda_5^5 = 0$. *Now, for* $\widehat{K}_2^t := (1-t)[e_2, e_5, e_7, v_5]^T + t[e_2, e_5, v_4, v_5]$ *at* $t = 0$,

$$\widehat{K}_2^0 X_{(257)}\Lambda_4^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ & x_2 & 1 \\ & x_3 & x_4 \\ & y & x_5 \\ \mathbf{0} & & x_6 \\ & & x_7 \\ & & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 1 & 0 \\ 0 & y & x_5 \\ 0 & 0 & x_7 \\ \star & \star & \star \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As assigned, $\lambda_2 = 1$ *and obviously* $\lambda_3 = 0$, *and the new variable* y *must be zero. And, since*

$$x_1\lambda_1 + 1 = 0,$$

$x_l = x_l^{(1)}$ *for* $l = 1, \ldots, 7$, *along with* $\lambda_1 = -\frac{1}{x_1^{(1)}}$, $\lambda_2 = 1$, $\lambda_3 = 0$, *is a solution of* $\widehat{K}_2^0 X_{(257)}\Lambda_4^4 = 0$. *Similarly, with* $\widehat{K}_2^1 = [e_2, e_5, v_4, v_5]^T$, *let* $x_1^{(2)}, \ldots, x_7^{(2)}, y^{(2)}, \lambda_1^{(2)}$,

$\lambda_2^{(2)}(=1), \lambda_3^{(2)}$ *be a solution of*

$$\widehat{K}_2^1 X_{(257)} \Lambda_4^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ & x_2 & 1 \\ & x_3 & x_4 \\ & y & x_5 \\ \text{\Large 0} & & x_6 \\ & & x_7 \\ & & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 1 & 0 \\ 0 & y & x_5 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then for $\widehat{K}_3^t := (1-t)[e_2, e_5, e_8, v_4, v_5]^T + t[v_1, v_2, v_3, v_4, v_5]$ *at* $t = 0$,

$$\widehat{K}_3^0 X_{[258]} \Lambda_3^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ & x_2 & 1 \\ & x_3 & x_4 \\ & y & x_5 \\ \text{\Large 0} & & x_6 \\ & & x_7 \\ & & z \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 1 & 0 \\ 0 & y & x_5 \\ 0 & 0 & z \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\lambda_3 = 1$, *as assigned, implies* $z = 0$. *On the other hand, since*

$$y\lambda_2 + x_5 = 0 \text{ and } x_1\lambda_1 + \lambda_2 = 0,$$

$x_l = x_l^{(2)}$ *for* $l = 1, \dots, 7$, $y = y^{(2)}$, $\lambda_2 = -\frac{x_5^{(2)}}{y^{(2)}}$, *and* $\lambda_1 = -\frac{\lambda_2}{x_1^{(2)}}$ *is a solution of* $\widehat{K}_3^0 X_{[258]} \Lambda_3^3 = 0$. □

Remark 2. In Example 8, $\widehat{K}_1^1$ defines a 5-plane L_3^1 containing L_3, $\widehat{K}_2^1$ defines a 4-plane L_3^2 containing L_3, and $\widehat{K}_3^1 = K_3$ represents L_3. So the strategy behind the homotopies we construct between intermediate nodes is the following. To find the 3-planes in the form of $X_{[258]}$ which meet L_1, L_2, L_3 (those $X_{[258]}$ are in $\sigma_3 \bullet \sigma_3 \bullet \sigma_3 \bullet \Omega(2,5,8)$), we first find the 3-planes $X_{(247)}$ which meet L_1, L_2, L_3^1 (those $X_{(247)}$ are in $\sigma_3 \bullet \sigma_3 \bullet \sigma_1 \bullet \Omega(2,4,7)$). Then we find the 3-planes $X_{(257)}$ which meet L_1, L_2, L_3^2 (those X's are in $\sigma_3 \bullet \sigma_3 \bullet \sigma_2 \bullet \Omega(2,5,7)$). Ultimately, we find the 3-planes $X_{[258]}$ which meet L_1, L_2, L_3.

4. Numerical results. An implementation of a previous version of the Pieri homotopy algorithm for numerical Schubert calculus [4] exists in the module of the extended version of PHCpack in [10] that also provides the SAGBI homotopy proposed in [3] for solving general problems in enumerative geometry numerically. In general, as reported in [4], the Pieri homotopy algorithms are much superior in speed as well as the range of applications than the SAGBI homotopies. We therefore compare only the results of the implementation of our algorithm with those of the code in PHCpack. All computations were carried out on a 400 MHz Intel Pentium II CPU with 256 MB of RAM, running on SunOS 5.6. In all the tables below #hty represents the total number of homotopies we followed in the corresponding cases, and **Wu** is the symbol representing our code.

1. $k_i = 1$ for all i, as shown in Table 1.

TABLE 1

m	p	**#soln**	#hty	**Wu**	**PHC**
3	2	**5**	21	220ms	720ms
4	2	**14**	63	1s270ms	5s50ms
3	3	**42**	183	13s420ms	41s480ms
5	2	**42**	195	8s870ms	39s870ms
6	2	**132**	360	13s330ms	1m13s
7	2	**429**	1196	1m16s	8m14s
4	3	**462**	1110	1m58s	8m59s
8	2	**1,430**	4,056	7m44s	53m2s
9	2	**4,862**	13,988	38m11s	6h29m1s
5	3	**6,006**	14,683	57m59s	7h53m28s
6	3	**87,516**	217,276	28h44m13s	-

The code in PHCpack requires a much bigger RAM than our code in all of the cases. For instance, for $(m, p) = (4, 3)$ above, PHC needs more than 7,044KB whereas **Wu** only needs 996KB.

2. $k_i > 1$ for certain i's, as shown in Tables 2–10. The first column of each table shows the numbers of all those k_i's.

TABLE 2
$(m, p) = (3, 2)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
321	**1**	6	40ms	470ms
222	**1**	6	60ms	550ms
2211	**2**	9	80ms	1s30ms
21111	**3**	13	130ms	2s290ms

TABLE 3
$(m, p) = (3, 3)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
333	**1**	9	160ms	2s250ms
3222	**1**	9	250ms	5s70ms
33111	**1**	9	200ms	4s420ms
32211	**2**	13	490ms	8s120ms
22221	**3**	21	870ms	10s480ms
222111	**6**	32	1s160ms	20s670ms
2211111	**11**	50	3s310ms	42s190ms
21111111	**21**	92	7s80ms	1m10s830ms

TABLE 4
$(m, p) = (4, 2)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
2222	**3**	20	220ms	6s750ms
3311	**2**	12	150ms	4s730ms
4211	**1**	8	60ms	3s70ms
32111	**3**	16	240ms	6s850ms
41111	**1**	8	80ms	2m510ms
221111	**6**	30	680ms	14s240ms
311111	**4**	20	380ms	8s970ms
2111111	**9**	41	910ms	16s460ms

TABLE 5
$(m, p) = (4, 3)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
44211	**1**	12	330ms	25s550ms
43311	**2**	16	750ms	53s700ms
43221	**2**	17	730ms	1m9s320ms
33222	**4**	29	1s730ms	1m39s800ms
222222	**16**	120	7s340ms	3m57s880ms
2222211	**26**	166	15s510ms	9m26s530ms
22221111	**45**	226	25s300ms	15m20s820ms
222111111	**79**	360	49s840ms	25m8s680ms
2211111111	**140**	622	1m35s740ms	41m42s700ms
21111111111	**252**	1,112	3m34s200ms	1h16m48s270ms

TABLE 6
$(m, p) = (5, 3)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
54321	**2**	20	1s240ms	6m0s660ms
44421	**3**	30	2s50ms	8m12s730ms
44322	**4**	37	3s340ms	9m23s770ms
43332	**5**	49	4s230ms	11m30s110ms
33333	**6**	65	5s80ms	10m21s130ms
543111	**3**	23	1s660ms	6m46s850ms
5421111	**4**	28	1s970ms	9m55s900ms
333321	**14**	118	12s540ms	24m29s30ms
3222222	**60**	451	1m2s180ms	1h14m22s370ms

TABLE 7
$(m, p) = (5, 2)$

$[k_1, \ldots, k_n]$	**#soln**	#hty	**Wu**	**PHC**
4222	**2**	16	270ms	24s560ms
5311	**1**	10	210ms	10s280ms
3322	**3**	23	360ms	35s40ms
22222	**6**	44	1s20ms	1m14s520ms

TABLE 8
$(m, p) = (5, 4)$

$[k_1, \ldots, k_n]$	#soln	#hty	Wu	PHC
44444	**1**	20	2s490ms	49m2s760ms
553322	**3**	33	6s480ms	1h48m10s470ms
443333	**9**	102	22s170ms	2h50m18s640ms
544322	**4**	42	7s950ms	2h14m59s90ms
4443221	**18**	145	45s710ms	-
4433222	**32**	261	1m25s430ms	-
2222222222	**3,396**	25,938	5h4m39s444ms	-

TABLE 9
$(m, p) = (6, 3)$

$[k_1, \ldots, k_n]$	#soln	#hty	Wu	PHC
333333	**40**	413	1m6s560ms	3h22m45s430ms
443322	**24**	208	30s920ms	-
433332	**30**	286	40s650ms	-
3333222	**104**	830	2m20s260ms	-
222222222	**876**	6,547	30m22s470ms	-

TABLE 10
$(m, p) = (6, 4)$

$[k_1, \ldots, k_n]$	#soln	#hty	Wu	PHC
664422	**3**	37	10s0ms	-
654333	**6**	70	22s850ms	-
554433	**10**	123	45s870ms	-
444444	**15**	220	1m10s440ms	22h58m54s70ms
33333333	**790**	8,413	1h15m45s778ms	>148.5h

As we can see from the results above, our novel approach of employing the Pieri homotopy algorithm for the numerical Schubert calculus has made a considerable advance in speed. And, in all the cases we have tried, the storage requirement for our code is much smaller than that of the existing code. The algorithm is particularly valuable when $k_i > 1$ appears.

REFERENCES

[1] W. V. D. HODGE AND D. PEDOE, *Methods of Algebraic Geometry*, Vol. II., Cambridge University Press, Cambridge, UK, 1968.

[2] S. KLEIMAN AND D. LAKSOV, *Schubert calculus*, Amer. Math. Monthly, 79 (1974), pp. 1061–1082.

[3] B. HUBER, F. SOTTILE, AND B. STURMFELS, *Numerical Schubert calculus*, J. Symbolic Comput., 26 (1998), pp. 767–788.

[4] B. HUBER AND J. VERSCHELDE, *Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control*, SIAM J. Control Optim., 38 (2000), pp. 1265–1287.

[5] T. Y. LI, T. SAUER, AND J. A. YORKE, *The cheater's homotopy: An efficient procedure for solving systems of polynomial equations*, SIAM J. Numer. Anal., 26 (1989), pp. 1241–1251.

[6] A. P. MORGAN AND A. J. SOMMESE, *Coefficient-parameter polynomial continuation*, Appl. Math. Comput., 29 (1989), pp. 123–160.

[7] A. P. MORGAN AND A. J. SOMMESE, *Errata: "Coefficient-parameter polynomial continuation"* [Appl. Math. Comput. 29 (1989), no. 2, part II, 123–160], Appl. Math. Comput., 51 (1992), p. 207.

[8] A. J. SOMMESE AND C. W. WAMPLER, *Numerical algebraic geometry*, in The Mathematics

of Numerical Analysis, Park City, Utah, 1995, J. Renegar, M. Shub, and S. Smale, eds., Lectures in Appl. Math. 32, AMS, Providence, RI, 1996, pp. 749–763.

[9] F. SOTTILE, *Pieri's formula via explicit rational equivalence*, Canad. J. Math., 49 (1997), pp. 1281–1298.

[10] J. VERSCHELDE, *Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation*, ACM Trans. Math. Software, 25 (1999), pp. 251–276.