HADAMARD INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

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Abstract. In this paper, we establish serveral inequalities of Hadamard's type for Wright-Convex functions.

1. Introduction

If $f:[a,b]\rightarrow\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2} \tag{1.1}$$

is known as the Hadamard inequality ([5]).

For some results which generalize, improve, and extend this famous integral inequality see [1] − [8], [10] − [15].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

Theorem 1. *If $f:[a,b]\rightarrow\mathbb{R}$ is a convex function, and H is defined on $[0,1]$ by*

$$H\left(t\right)=\frac{1}{b-a}\int_a^b f\left(tx+\left(1-t\right)\frac{a+b}{2}\right)dx, \tag{1.2}$$

then H is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$f\left(\frac{a+b}{2}\right)=H\left(0\right)\leq H\left(t\right)\leq H\left(1\right)=\frac{1}{b-a}\int_a^b f\left(x\right)dx \tag{1.3}$$

In [10], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1).

Theorem 2. *If $f:[a,b]\rightarrow\mathbb{R}$ is a convex function, and F is defined on $[0,1]$ by*

$$\begin{aligned}F\left(t\right)=\frac{1}{2\left(b-a\right)}\int_a^b \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)&\\ +f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\Bigg]dx&,\end{aligned} \tag{1.4}$$

then F is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f\left(x\right)dx=F\left(0\right)\leq F\left(t\right)\leq F\left(1\right)=\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{1.5}$$

We recall the definition of a Wright-convex function:

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Definition 1. *(see* [9, p. 223]*). We say that* $f:[a,b]\to\mathbb{R}$ *is a Wright-convex function, if, for all* $x,\ y+\delta\in[a,b]$ *with* $x<y$ *and* $\delta\geq 0$*, we have*

$$f(x+\delta)+f(y)\leq f(y+\delta)+f(x). \tag{1.6}$$

Let $C([a,b])$ be the set of all convex functions on $[a,b]$ and $W([a,b])$ be the set of all Wright-convex functions on $[a,b]$. Then $C([a,b])\subsetneqq W([a,b])$. That is, a convex function must be a Wright-convex function but not conversely (see [9, p. 224]).

In this paper, we shall establish several inequalities of Hadamard's type for Wright-convex functions.

2. Main Results

In order to prove our main theorems, we need the following lemma:

Lemma 1. *If* $f:[a,b]\to\mathbb{R}$*, then the following statements are equivalent:*

(1) $f\in W([a,b])$;
(2) *for all* $s,t,u,v\in[a,b]$ *with* $s\leq t\leq u\leq v$ *and* $t+u=s+v$*, we have*

$$f(t)+f(u)\leq f(s)+f(v). \tag{2.1}$$

Proof. Suppose $f\in W([a,b])$. If $s,t,u,v\in[a,b]$, and $s\leq t\leq u\leq v$, where $t+u=s+v$, then we can write $x=s$, $x+\delta=t$, $y=u$, $y+\delta=v$, it follows from (1.6) that

$$f(t)+f(u)\leq f(s)+f(v).$$

Conversely, if $x,y+\delta\in[a,b]$, $x<y$ and $\delta\geq 0$. We may have

$$x\leq x+\delta\leq y\leq y+\delta$$

or

$$x\leq y\leq x+\delta\leq y+\delta.$$

In either case we have, by (2.1), that

$$f(x+\delta)+f(y)\leq f(x)+f(y+\delta).$$

Thus $f\in W([a,b])$. ∎

Theorem 3. *Let* $f\in W([a,b])\cap L_1[a,b]$*. Then (1.1) holds.*

Proof. For (2.1), we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \frac{1}{(b-a)}\int_a^{\frac{a+b}{2}}\left[f\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)\right]dx\\
&\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]dx \quad \left(x\leq\frac{a+b}{2}\leq\frac{a+b}{2}\leq a+b-x\right)\\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}f(x)\,dx+\int_{\frac{a+b}{2}}^b f(x)\,dx\right]\\
&= \frac{1}{b-a}\int_a^b f(x)\,dx,
\end{aligned}$$

and

$$\begin{aligned}\frac{f(a)+f(b)}{2} &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} [f(a)+f(b)]\,dx \\ &\geq \frac{1}{b-a}\int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,dx \quad (a\leq x\leq a+b-x\leq b) \\ &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)\,dx\right] \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx,\end{aligned}$$

This completes the proof. ∎

Theorem 4. *Let $f \in W([a,b]) \cap L_1[a,b]$ and let H be defined as in (1.2). Then $H \in W([0,1])$ is increasing on $[0,1]$, and (1.3) holds for all $t \in [0,1]$.*

Proof. If $s,t,u,v \in [0,1]$ and $s \leq t \leq u \leq v$, $t+u = s+v$, then for $x \in \left[a, \frac{a+b}{2}\right]$ we have

$$\begin{aligned} b &\geq sx + (1-s)\frac{a+b}{2} \\ &\geq tx + (1-t)\frac{a+b}{2} \\ &\geq ux + (1-u)\frac{a+b}{2} \\ &\geq vx + (1-v)\frac{a+b}{2} \geq a, \end{aligned}$$

and if $x \in \left[\frac{a+b}{2}, b\right]$, then

$$\begin{aligned} a &\leq sx + (1-s)\frac{a+b}{2} \\ &\leq tx + (1-t)\frac{a+b}{2} \\ &\leq ux + (1-u)\frac{a+b}{2} \\ &\leq vx + (1-v)\frac{a+b}{2} \leq b, \end{aligned}$$

where

$$\begin{aligned} &\left[tx + (1-t)\frac{a+b}{2}\right] + \left[ux + (1-u)\frac{a+b}{2}\right] \\ &\qquad = \left[sx + (1-s)\frac{a+b}{2}\right] + \left[vx + (1-v)\frac{a+b}{2}\right]. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(ux + (1-u)\frac{a+b}{2}\right) \\ &\qquad \leq f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(vx + (1-v)\frac{a+b}{2}\right). \end{aligned}$$

for all $x \in [a, b]$. Integrating this inequality over x on $[a, b]$, and dividing both sides by $b - a$, yields

$$H(t) + H(u) \le H(s) + H(v).$$

Hence, $H \in W([0, 1])$.

Next, if $0 \le s \le t \le 1$ and $x \in \left[a, \frac{a+b}{2}\right]$, then

$$\begin{aligned} tx + (1-t)\frac{a+b}{2} &\le sx + (1-s)\frac{a+b}{2} \\ &\le s(a+b-x) + (1-s)\frac{a+b}{2} \\ &\le t(a+b-x) + (1-t)\frac{a+b}{2}, \end{aligned}$$

where

$$\begin{aligned} &\left[sx + (1-s)\frac{a+b}{2}\right] + \left[s(a+b-x) + (1-s)\frac{a+b}{2}\right] \\ &\qquad = \left[tx + (1-t)\frac{a+b}{2}\right] + \left[t(a+b-x) + (1-t)\frac{a+b}{2}\right]. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} H(s) &= \frac{1}{b-a}\int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right)dx \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left[f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right)\right]dx \\ &\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right]dx \\ &= \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx \\ &= H(t). \end{aligned}$$

Thus, H is increasing on $[0, 1]$, and (1.3) holds for all $t \in [0, 1]$.

This completes the proof. ∎

Theorem 5. *Let $f \in W([a, b]) \cap L_1[a, b]$ and let F be defined as in (1.4). Then $F \in W([0, 1])$ is increasing on $[0, 1]$, and (1.5) holds for all $t \in [0, 1]$.*

Proof. If $s, t, u, v \in [0, 1]$ and $s \le t \le u \le v$, $t + u = s + v$, then

$$\begin{aligned} a &\le \left(\frac{1+v}{2}\right)a + \left(\frac{1-v}{2}\right)x \\ &\le \left(\frac{1+u}{2}\right)a + \left(\frac{1-u}{2}\right)x \\ &\le \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x \\ &\le \left(\frac{1+s}{2}\right)a + \left(\frac{1-s}{2}\right)x \le b \text{ for all } x \in [a, b], \end{aligned}$$

and

$$\begin{aligned} a &\leq \left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x \\ &\leq \left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) x \\ &\leq \left(\frac{1+u}{2}\right) b+\left(\frac{1-u}{2}\right) x \\ &\leq \left(\frac{1+v}{2}\right) b+\left(\frac{1-v}{2}\right) x \leq b \text{ for all } x \in [a,b], \end{aligned}$$

where

$$\begin{aligned} &\left[\left(\frac{1+u}{2}\right) a+\left(\frac{1-u}{2}\right) x\right]+\left[\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x\right] \\ &\qquad =\left[\left(\frac{1+v}{2}\right) a+\left(\frac{1-v}{2}\right) x\right]+\left[\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) x\right]+\left[\left(\frac{1+u}{2}\right) b+\left(\frac{1-u}{2}\right) x\right] \\ &\qquad =\left[\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x\right]+\left[\left(\frac{1+v}{2}\right) b+\left(\frac{1-v}{2}\right) x\right]. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &f\left(\left(\frac{1+u}{2}\right) a+\left(\frac{1-u}{2}\right) x\right)+f\left(\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x\right) \\ &\quad +f\left(\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) x\right)+f\left(\left(\frac{1+u}{2}\right) b+\left(\frac{1-u}{2}\right) x\right) \\ &\quad \leq f\left(\left(\frac{1+v}{2}\right) a+\left(\frac{1-v}{2}\right) x\right)+f\left(\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x\right) \\ &\qquad +f\left(\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x\right)+f\left(\left(\frac{1+v}{2}\right) b+\left(\frac{1-v}{2}\right) x\right), \end{aligned}$$

for all $x \in [a,b]$. Integrating this inequality over x on $[a,b]$, and dividing both sides by $2(b-a)$, we have

$$F(t)+F(u) \leq F(s)+F(v),$$

hence, $F \in W([0,1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in [a,b]$, then

$$\begin{aligned} \left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x &\leq \left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x \\ &\leq \left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) (a+b-x) \\ &\leq \left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) (a+b-x), \end{aligned}$$

and

$$\begin{aligned}\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)(a+b-x) &\leq \left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)(a+b-x)\\ &\leq \left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\\ &\leq \left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\end{aligned}$$

where

$$\begin{aligned}&\left[\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right]+\left[\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)(a+b-x)\right]\\ &\qquad=\left[\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right]+\left[\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)(a+b-x)\right],\end{aligned}$$

and

$$\begin{aligned}&\left[\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)(a+b-x)\right]+\left[\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right]\\ &\qquad=\left[\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)(a+b-x)\right]+\left[\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right].\end{aligned}$$

Thus

$$\begin{aligned}F(s) &= \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right)\right.\\ &\qquad \left.+f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)\right]dx\\ &= \frac{1}{4(b-a)}\int_a^b \left\{\left[f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right)\right.\right.\\ &\qquad \left.+f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)(a+b-x)\right)\right]\\ &\qquad +\left[f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)(a+b-x)\right)\right.\\ &\qquad \left.\left.+f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)\right]\right\}dx\\ &\leq \frac{1}{4(b-a)}\int_a^b \left\{\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.\right.\\ &\qquad \left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right]\\ &\qquad +\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right.\\ &\qquad \left.\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]\right\}dx.\end{aligned}$$

Hence, F is increasing on $[0,1]$ and (1.5) holds for all $t\in[0,1]$.

This completes the proof. ∎

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