EFFECT OF SAMPLING ERRORS ON ARRAY GAIN

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Abstract

The "Bryn Processor" is a well-known array processor; its design depends on spectral densities of the signal and noise processes. Array output signal-to-noise ratio can be decreased because of errors in sampling times: (1) When array design parameters are estimated from sampled data; (2) when operating on sampled input data. We model errors in sampling times as a discrete-parameter random process. Expressions for array output SNR of the Bryn processor are determined for the two situations listed above. Numerical results are presented for some specific assumptions on signal, noise, and sampling error.

Introduction

This paper contains an analysis of the effect of sampling time jitter on the array output signal-to-noise ratio and array gain of the "Bryn processor". This is a well-known processor [1], [2] for the detection of a stationary band-limited Gaussian signal in independent stationary band-limited Gaussian noise. The design of the processor is data-dependent, varying with the spectral properties of signal and noise.

Errors in sampling time can arise in either the evaluation data (data being tested for signal) or in the design data (the data used to determine the components of the processor). Thus, one could have sampling jitter in the evaluation data and jitter in the design data, or jitter in evaluation data and no jitter in design data, or jitter in design data and no jitter in the evaluation data. Each of these combinations is considered here.

We proceed as follows. First, the digitized Bryn processor is derived, assuming no jitter. The processor is then implemented with an array followed by a spectral filter. Assuming this implementation, we compute array output SNR for the various combinations of jitter in design data and evaluation data. The resulting expressions for array SNR are then analyzed to determine the effect of jitter. Finally, we present curves of array output SNR as a function of frequency, under two sets of assumptions on signal, noise, and jitter characteristics.

Our results are stated in terms of array output SNR. Array gain is usually defined as the ratio of output SNR to input SNR. Thus, the comparisons presented here for output SNR can be interpreted as comparisons of array gain.

Derivation and Implementation of the Bryn Processor

The basic philosophy of the Bryn processor is given in [1], and its implementation (for continuous-time operation) under the assumptions to be used here is given in [2]. We give the derivation of the digitized processor, and its implementation; this will also help to fix our definitions and framework.

A set of K sensors is assumed; the output of the i^{th} sensor is $\ell_i(t)$, $0 \leq t \leq T_0$. $L(t)$ is the K-component vector with $\ell_i(t)$ as i^{th} component. If noise alone is present, $\ell_i(t) = n_i(t)$; if signal-plus-noise is present, $\ell_i(t) = s_i(t) + n_i(t)$. The following assumptions will be made. (1) Signal and noise are segments of independent and stationary zero-mean Gaussian processes, with spectral densities $S(\omega)$ (for signal) and $N(\omega)$ (for noise), the spectral density being the same for each sensor. (2) Signal is present at one sensor only if signal is present at all sensors. (3) The signal and noise spectral densities are essentially band-limited, with the frequency band having positive frequencies in the region (f_ℓ, f_u). (4) In implementing the processor, the signal will be assumed to be a plane wave, so that $s_i(t) = s(t - \tau_i)$, where τ_i is a time delay measured from some reference. (5) The vector of sensor outputs, $L(t)$, can be represented by

$$L(t) = \sum_{k=-f_u T_0}^{f_u T_0} Z(k\Omega) e^{jk\Omega t}, \quad 0 \leq t \leq T_0 \qquad (1)$$

where $\Omega = 2\pi/T_0$, and $Z(k\Omega)$ is a K-component column vector with i^{th} component $Z_i(k\Omega)$. In order for equation (1) to accurately represent $L(t)$, one must have either $f_u \to \infty$ or $T_0 \to \infty$. We assume here (as in [1], [2]) that T_0 is sufficiently large, so that equation (1) adequately represents $L(t)$. Further, since $Z(k\Omega)$ is a Fourier coefficient, $Z^*(k\Omega) = Z(-k\Omega)$. (6) $Z_i(k\Omega) \equiv Y_i(k\Omega)$ when noise alone is present; $Z_i(k\Omega) \equiv X_i(k\Omega) + Y_i(k\Omega)$ when signal-plus-noise is present. We assume that X_i and Y_i can be adequately approximated by a discrete Fourier transform. Thus

$$X_i(k\Omega) \equiv (1/\sqrt{M}) \sum_{m=0}^{M-1} s_i(ma) e^{-jk\Omega ma} \qquad (2)$$

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$$Y_i(k\Omega) \equiv 1/\sqrt{M} \sum_{m=0}^{M-1} n_i(ma)e^{-jk\Omega ma} \tag{3}$$

where $a \leq 1/2B$ is the nominal sampling interval and $M=T_0/a$ is the total number of samples taken.

(7) The Fourier coefficients $\{Z_i(k\Omega), f_\ell T_0 \leq k \leq f_u T_0\}$ are a set of independent random variables for each $i = 1,\ldots,K$. If one had $T_0 \to \infty$ and equation (3) were actually a Fourier transform rather than an approximation by a DFT, this would necessarily follow from the fact that $s_i(t)$ and $n_i(t)$ are Gaussian processes [4].

To optimally detect the signal, we form the likelihood ratio [4]. Since all of the information in the data is contained (under the above assumptions) in the positive-frequency Fourier coefficients, this involves taking the ratio of the joint probability density function of the positive-frequency Fourier coefficients under the hypothesis of signal present to the ratio of the joint probability density function under the hypothesis that signal is absent. Since the Fourier coefficients are assumed independent for different values of $k\Omega$, and the data is Gaussian, the likelihood ratio becomes

$$\Lambda(Z) = \gamma \exp\{-1/2 \sum_{k=f_\ell T_0}^{f_u T_0} [Z^{*T}(k\Omega)\{(\mathrm{cov}[X(k\Omega) + Y(k\Omega)])^{-1} - (\mathrm{cov}[Y(k\Omega)])^{-1}\}Z(k\Omega)]\} \tag{4}$$

where γ is a constant, and $\mathrm{cov}(\cdot)$ denotes the covariance matrix.

Let $Q(k\Omega)$ be the matrix of normalized correlation coefficients of the vector noise process, with ij element $q_{ij}(k\Omega)$. $N(k\Omega)q_{ij}(k\Omega)$ is approximately (large M) the cross-spectral density for the noise process in the i^{th} and j^{th} channels; $q_{ij}(k\Omega) = E\, Y_i(k\Omega)Y_j^*(k\Omega)/N(k\Omega)$. The covariance matrix of the vector noise process is thus $N(k\Omega)Q(k\Omega)$, while the covariance matrix of the plane wave vector signal process is $S(k\Omega)V(k\Omega)V^{*T}(k\Omega)$, where $V(k\Omega)$ is a K-component vector whose i^{th} component is $e^{jk\Omega\tau_i}$ (see [3] for derivation of the covariance matrices). We assume that $Q^{-1}(k\Omega)$ exists for all frequencies $k\Omega$ in the data band.

If $\Lambda(Z)$ represents the likelihood ratio for the Fourier coefficients $\{Z(k\Omega), f_\ell T_0 \leq k \leq f_u T_0\}$, then one has [3]

$$-2 \log \Lambda(Z) = \sum_{k=f_\ell T_0}^{f_u T_0} \frac{S(k\Omega)|Z^{*T}(k\Omega)W(k\Omega)|^2}{B(k\Omega)} \tag{5}$$

where $W(k\Omega) = Q^{-1}(k\Omega)V(k\Omega)$, and $B(k\Omega) = N(k\Omega)[N(k\Omega) + S(k\Omega)V^{*T}(k\Omega)Q^{-1}(k\Omega)V(k\Omega)]$. The processor can be implemented as shown in Figure 1, where the Fast Fourier Transform in the i^{th} channel computes $Z_i(k\Omega) = 1/\sqrt{M}\sum_{m=0}^{M-1} \ell_i(ma)e^{-jk\Omega ma}$, and the spectral filter is $F(k\Omega) = [S(k\Omega)/B(k\Omega)]^{1/2}$. The output of the spectral filter is passed through a square-law device, and then summed over $k = f_\ell T_0,\ldots,f_u T_0$. This sum is $-2\log\Lambda(Z)$, under the above assumptions, and is compared with a threshold.

We assume hereafter that the processor is implemented as shown in Figure 1; the array is that part of the processor preceding the spectral filter $F(k\Omega)$.

Bryn Processor with Sampling Jitter

Suppose now that the actual sampling times are $\{ma + \varepsilon_m\}$, where $\varepsilon_m \equiv \varepsilon(ma)$ represents the jitter for m^{th} sample. $\{\varepsilon_m, 0 \leq m \leq M-1\}$ is assumed to be a random process independent of both signal and noise. ε_m is also assumed to be independent of ε_n for $m \neq n$, and the probability distribution function of ε_m is the same as that of ε_n, with characteristic function $\Phi(\omega) = E\, e^{j\omega\varepsilon_m}$. The output of the FFT for the i^{th} array element at angular frequency $k\Omega$ is then $[Z_i(k\Omega)]_{jitter} = 1/\sqrt{M}\sum_{m=0}^{M-1} \ell_i(ma + \varepsilon_m)e^{-jk\Omega ma}$. The covariance matrices of the jittered signal and noise processes are derived in [3]. For the jittered signal one obtains $E[X(k\Omega)X^{*T}(k\Omega)]_J = D + |\Phi(k\Omega)|^2 S(k\Omega)V(k\Omega)V^{*T}(k\Omega)$ where $D = [d_{ij}]$, $d_{ij} = (1/2\pi)\int_{Band} e^{j\omega(\tau_i - \tau_j)}[1 - |\Phi(\omega)|^2]S(\omega)d\omega$. For the jittered noise, the covariance matrix is $E[Y(k\Omega)Y^{*T}(k\Omega)]_J = U + N(k\Omega)|\Phi(k\Omega)|^2Q(k\Omega)$, where $U = [u_{ij}]$, $u_{ij} = (1/2\pi)\int_{Band}[1 - |\Phi(\omega)|^2]q_{ij}(\omega)N(\omega)d\omega$. The spectral densities for the jittered signal and noise thus become $S_J(k\Omega) = (1/2\pi)\int_{Band}[1 - |\Phi(\omega)|^2]S(\omega)d\omega + |\Phi(k\Omega)|^2 S(k\Omega)$, $N_J(k\Omega) = (1/2\pi)\int_{Band}[1 - |\Phi(\omega)|^2]N(\omega)d\omega + |\Phi(k\Omega)|^2 N(k\Omega)$. The normalized correlation matrix for the noise is now $Q_J(k\Omega) = (u_{11} + N(k\Omega)|\Phi(k\Omega)|^2)^{-1}[U + N(k\Omega)|\Phi(k\Omega)|^2Q(k\Omega)]$.

In computing array output SNR, under the assumption that the design data contains jitter, one has $[W(k\Omega)]_J = Q_J^{-1}(k\Omega)V(k\Omega)$.

Expressions for Array Signal-to-Noise Ratio

The array output SNR at angular frequency k is defined as the ratio of the difference in the average power output when signal is present and signal is absent, to the average power output when signal is absent, or

$$\rho(k\Omega) = \frac{E\{|W^{*T}(k\Omega)[X(k\Omega)+Y(k\Omega)]|^2 - |W^{*T}(k\Omega)Y(k\Omega)|^2\}}{E|W^{*T}(k\Omega)Y(k\Omega)|^2} = \frac{W^{*T}(k\Omega)\{E[X(k\Omega)X^{*T}(k\Omega)]\}W(k\Omega)}{W^{*T}(k\Omega)\{E[Y(k\Omega)Y^{*T}(k\Omega)]\}W(k\Omega)}. \tag{6}$$

For the various combinations of jitter, we obtain the following results for array SNR. For the sake of brevity, we assume $\tau_i = 0$, $i = 1,\ldots,K$. For $\tau_i \neq 0$, see [3]. When $\tau_i = 0$, $V_i(k\Omega) = V_i(0) = 1$, $i = 1,\ldots,K$.

Case 1. No jitter in design data, no jitter in evaluation data.

$$\rho_1(k\Omega) = \frac{S(k\Omega)}{N(k\Omega)} V^{*T}(0)\, Q^{-1}(k\Omega)V(0) \tag{7}$$

Case 2. Jitter in design data and jitter (with same statistical properties) in evalution data.

$$\rho_2(k\Omega) = \frac{d_{11} + |\Phi(k\Omega)|^2 S(k\Omega)}{u_{11} + |\Phi(k\Omega)|^2 N(k\Omega)} V^{*T}(0) Q_J^{-1}(k\Omega) V(0) \quad (8)$$

Case 3. Jitter in evaluation data; no jitter in design data.

$$\rho_3(k\Omega) = \frac{[d_{11}+|\Phi(k\Omega)|^2 S(k\Omega)][V^{*T}(0)Q^{-1}(k\Omega)V(0)]^2}{V^T(0)Q_J^{-1}(k\Omega)[U+|\Phi(k\Omega)|^2 N(k\Omega)Q(k\Omega)]Q_J^{-1}(k\Omega)V(0)} \quad (9)$$

Case 4. Jitter in design data; no jitter in evaluation data.

$$\rho_4(k\Omega) = \frac{S(k\Omega)}{N(k\Omega)} \frac{[V^T(0)Q_J^{-1}(k\Omega)V(0)]^2}{V^T(0)Q_J^{-1}(k\Omega)Q(k\Omega)Q_J^{-1}(k\Omega)V(0)} \quad (10)$$

Effect of Jitter

The effect of sampling jitter on array output SNR and array gain can be analyzed by examining the above expressions for output SNR.

First, suppose that $U = \alpha_k Q(k\Omega)$. This will occur, for example, when the noise is independent between channels. More generally, it will occur whenever the noise cross-spectral density between the i^{th} and j^{th} channels equals $N(k\Omega)f[|i-j|]$, for $i,j = 1,\ldots,K$. Since $q_{11} = 1$, α_k must equal u_{11}, and since $Q_J(k\Omega) = (u_{11} + N(k\Omega)|\Phi(k\Omega)|^2)^{-1}$ $(U + N(k\Omega)|\Phi(k\Omega)|^2 Q(k\Omega))$, one has $Q_J(k\Omega) = Q(k\Omega)$. Examining expressions (8) and (10) for the array SNR when the design data contains jitter, one sees that $\rho_1(k\Omega) = \rho_4(k\Omega)$ and $\rho_2(k\Omega) = \rho_3(k\Omega)$, so that in this case jitter in the design data has no effect on output SNR. However, it can be expected that jitter in the design data will have an effect on detection performance, since the quantities in both the array and the spectral filter are changed when there is jitter in the design data; also, the processor loses its interpretation as an approximation to the likelihood ratio when jitter is present [3].

If the noise is independent between channels at frequency $k\Omega$ (so that $\rho_1(k\Omega) = \rho_4(k\Omega)$ and $\rho_2(k\Omega) = \rho_3(k\Omega)$), one has $Q(k\Omega) = I_k$ (identity matrix in E^k). Thus, $\rho_1(k\Omega) = KS(k\Omega)/N(k\Omega)$. The array output SNR when the evaluation data contains jitter is given by

$$\rho_2(k\Omega) = \frac{K[(1/2\pi)\int_{Band}[1-|\Phi(\omega)|^2]S(\omega)d\omega+S(k\Omega)|\Phi(k\Omega)|^2)}{(1/2\pi)\int_{Band}[1-|\Phi(\omega)|^2]N(\omega)d\omega+N(k\Omega)|\Phi(k\Omega)|^2} \quad (11)$$

Thus, if $\Phi(\omega) \cong 1$ across the data bandwidth, then $\rho_1(k\Omega) \cong \rho_2(k\Omega)$. However, if $\Phi(\omega) \cong 0$ across the data bandwidth, then $\rho_2(k\Omega) \cong K \int_{Band} S(\omega)d\omega / \int_{Band} N(\omega)d\omega$. In this case, there will be a loss in peak array SNR, and a complete loss of frequency selectivity.

If the noise is not independent between channels, but $U = u_{11}Q(k\Omega)$, then the behavior of $\rho_1(k\Omega)/\rho_2(k\Omega)$ is the same as if the noise were independent between channels. If it is not true that $U = u_{11}Q(k\Omega)$, the effect of jitter is not so easy to analyze [3].

Since $\Phi(\omega)$ is an L_1 Fourier transform, $|\Phi(\omega)| \to 0$ as $|\omega| \to \infty$. Thus, the effect of sampling jitter on array SNR and array gain will usually become more serious as the high-frequency content of the data increases.

If one is forced to work with a system having a fixed jitter characteristic, and the resulting array gain and detection performance is unsatisfactory, the effect of jitter may be reduced by sampling more frequently. This is because the effect of jitter is contained in its characteristic function $\Phi(\omega)$. If the sampling interval is decreased, then the jitter characteristic function will represent a density function confined to a smaller region, and so $\Phi(\omega)$ would go to zero more slowly. This will be the case if the shape of the jitter density function is independent of the width of the sampling interval. This is illustrated in the numerical examples.

Numerical Results

Several calculations of array gain were made for specific assumptions on the characteristics of the signal, noise, and jitter. We present here curves for two sets of assumptions. Additional numerical results are contained in [3].

The jitter was assumed to have triangular density function $p(\varepsilon)$ of width a (the sampling interval). Thus

$$\begin{aligned} p(\varepsilon) &= (2/a)^2(a/2 - |\varepsilon|) \quad |\varepsilon| < a/2 \\ &= 0 \qquad\qquad\qquad\quad |\varepsilon| \geq a/2 \ . \end{aligned}$$

In Figures 2 and 3, the data frequency is normalized by dividing by the bandwidth B. λ_0 is the normalized center frequency. The array SNR ρ in Figure 2 is computed for only one channel; since the data presented in Figure 2 is independent between channels, the value of ρ for one channel is the same as ρ/K for K channels. Thus, the ratio of ρ_1(no jitter) to ρ_2(jitter) is the same for one channel as for K channels. Also, since the data spectral densities are symmetric about λ_0 (in the positive-frequency region), we present only the array SNR as a function of frequency from $\lambda = \lambda_0$ to $\lambda = \lambda_0 + 1/2$ (i.e., over half the data band).

The results of Figure 2 are for white noise (in the data band) with spectral density $(\sqrt{2}/\pi)P_S$, and for signal with spectral density $(2\sqrt{2}/\pi)P_s$ $(1 + 2[\lambda - \lambda_0]^4)^{-1}$. The sampling interval is $a = (2B)^{-1}$. The loss in peak array SNR (at $\lambda = \lambda_0$) is about 13.5%, or .63 db. The array has a complete loss of frequency selectivity at $\lambda_0 = 4$ (center frequency is equal to 4B).

Figure 3 shows a much more serious effect. This figure represents a signal with bimodal spectral density (in the positive-frequency region) given by

$$S(\lambda) = \frac{10\sqrt{2}}{\pi} P_S \left[\frac{1}{1+[20(\lambda-\lambda_1)]^4} + \frac{1}{1+[20(\lambda-\lambda_2)]^4}\right]$$

$$N(\lambda) = \frac{2}{\pi} P_S \left[\frac{1}{1+[4(\lambda-\lambda_3)]^2} + \frac{1}{1+[4(\lambda-\lambda_4)]^2}\right] + (1/10) P_S$$

where $\lambda_1 = \lambda_0 - 1/8$, $\lambda_2 = \lambda_0 + 1/8$, $\lambda_3 = \lambda_0 - 1/4$, and $\lambda_4 = \lambda_0 + 1/4$. Thus, the noise spectral density peaks are at slightly different frequencies than those for the signal, and the noise envelopes are wider. This data is for a 4-channel array, and the noise was assumed to have cross-spectral density $N_{ij}(\lambda)$ given by $N_{ij}(\lambda) = \frac{1}{|j-i+1|} [N(\lambda) - (1/10)P_S]$, $i \neq j$. The numerical results are presented only for ρ_1 and ρ_2. Although the matrix U does not satisfy the relation $U = u_{11}\, Q(k\Omega)$ across the data band because of the white noise component, the relatively small power in the white noise component (compared to that of the colored noise component) results in $\rho_1 \cong \rho_4$ and $\rho_2 \cong \rho_3$. The sampling interval is $a = (8B)^{-1}$. The curves show a complete loss in frequency selectivity at $\lambda_0 = 16$, and a maximum loss in array output SNR of approximately 75%, or 6 db.

If the sampling interval a is decreased, then the effect is to increase the value of λ_0 at which the array loses frequency selectivity. Thus, if the data of Figure 3 were used for a sampling interval of $(32B)^{-1}$, rather than $(8B)^{-1}$, the array would not lose all frequency selectivity until $\lambda_0 = 64$. This is because the characteristic function would change from $\Phi(\lambda) = \text{sinc}^2(\pi\lambda/16)$ to $\Phi(\lambda) = \text{sinc}^2(\pi\lambda/64)$. In addition, the decrease in peak array SNR would be reduced at lower values of λ_0; for example, at $\lambda_0 = 16$, the peak array SNR would be reduced by only 13% (rather than 75%).

The loss of 6 db would be serious. However, the fact that such losses can occur seems more important than the actual values obtained here, since these values are for specific assumptions on the signal, noise, and jitter. It seems clear that one should attempt to determine the jitter characteristics and their effect for any array, using appropriate representations of signal and noise. The effect of jitter can usually be assumed to become more serious as the high-frequency content of the data increases. However, the effect can be alleviated in some cases by sampling more frequently.

Acknowledgement

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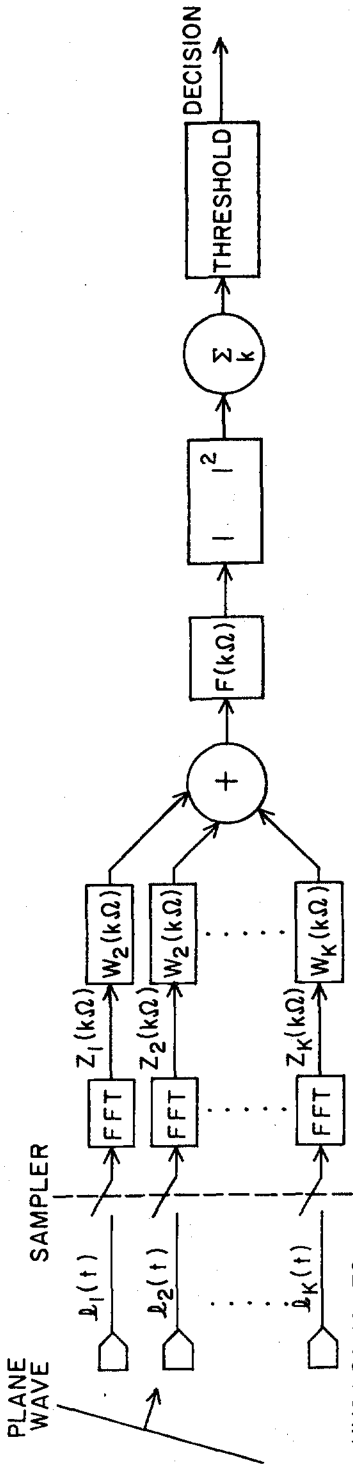

Fig. 1. Bryn Processor

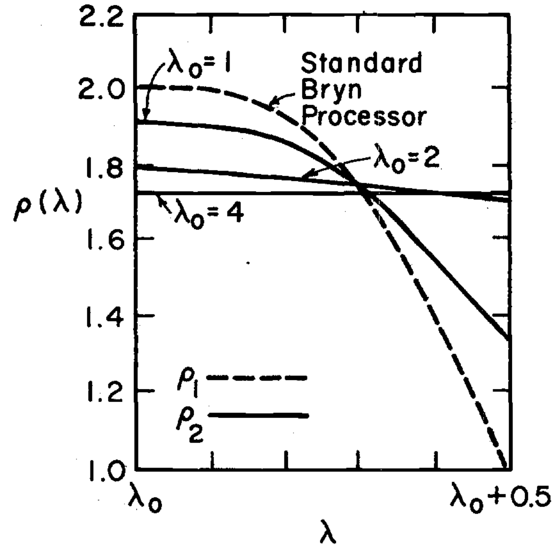

Fig. 2. Array Output SNR

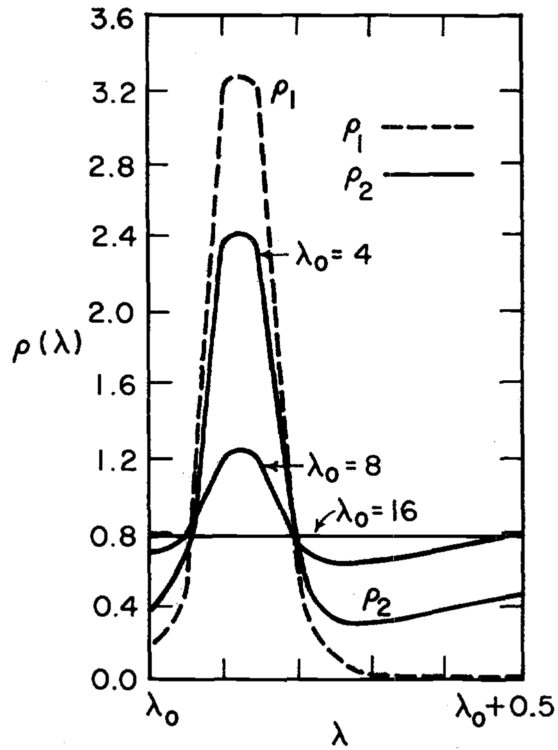

Fig. 3. Array Output SNR