Planar Dirac electron in Coulomb and magnetic fields: A Bethe ansatz approach

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The Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields is an example of the so-called quasi-exactly solvable models. The solvable parts of its spectrum were previously solved from the recursion relations. In this work we present a purely algebraic solution based on the Bethe ansatz equations. It is realized that, unlike the corresponding problems in the Schrödinger and the Klein–Gordon cases, here the unknown parameters to be solved for in the Bethe ansatz equations include not only the roots of the wave function assumed, but also a parameter from the relevant operator. We also show that the quasi-exactly solvable differential equation does not belong to the classes based on the algebra sl_2. © 2002 American Institute of Physics.
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I. INTRODUCTION

Recently a new type of spectral problem, the so-called quasi-exactly solvable model (QESM), was discovered by physicists and mathematicians.[1–8] This is a special class of quantum-mechanical problems for which analytical solutions are possible only for parts of the energy spectra and for particular values of the fundamental parameters. The reason for such quasi-exactly solvability is usually the existence of a hidden Lie-algebraic structure.[2–6] More precisely, a quasi-exactly solvable (QES) Hamiltonian can be reduced to a quadratic combination of the generators of a Lie group with finite-dimensional representations.

The first physical example of QESM in atomic physics is the system of two electrons moving in an external oscillator potential discussed in Refs. 9 and 10. The authors of these works apparently were unaware of the mathematical development in QESM. Later, several physical QESMs were discovered, which include the two-dimensional Schrödinger,[11] the Klein–Gordon,[12] and the Dirac equations[13] of an electron moving in an attractive/repulsive Coulomb field and a homogeneous magnetic field. The essential features shared by all these above examples are as follows. The differential equations are solved according to the standard procedure. After separating out the asymptotic behaviors of the system, one obtains an equation for the part which can be expanded as a power series of the basic variable. But instead of the two-step recursion relations for the coefficients of power series so often encountered in exactly solvable problems, one gets three-step recursion relations. The complexity of the recursion relations does not allow one to determine the energy spectrum exactly from the normalizability of the eigenfunctions. However, one can impose a sufficient condition for normalizability by terminating the series at a certain order of power of the variable; i.e., by choosing a polynomial. By doing so one could obtain exact solutions to the original problem, but only for certain energies and for specific values of the parameters of the problem. These parameters, namely, are the frequency of the oscillator potential and the external magnetic fields.

In Ref. 14 a systematic and unified algebraic treatment was given to the above-mentioned systems, with the exception of the Dirac case. This was made possible by realizing that these

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systems are governed essentially by the same basic equation, which is quasi-exactly solvable owing to the existence of a hidden sl_2 algebraic structure. This algebraic structure was first realized by Turbiner for the case of two electrons in an oscillator potential.[15] In this algebraic approach, analytic expressions of the solvable parts of the energy spectrum and the allowed parameters were expressible in terms of the roots of a set of Bethe ansatz equations.

In this article we would like to extend the method of Ref. 14 to the planar Dirac equation of an electron in the Coulomb and magnetic fields. It turns out that a set of Bethe ansatz equations can also be set up in this case. However, unlike the systems considered in Ref. 14, here the unknown variables in the Bethe ansatz equations involved not only the roots of the wave functions assumed, but also a parameter from the relevant operator. We also demonstrate that the Bethe ansatz approach yields the same spectrum as that obtained by solving recursion relations. Finally, we show that the quasi-exactly solvability of this system is not related to the sl_2 algebra.

II. THE DIRAC EQUATION

In $2+1$ dimensions the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g^{\mu\nu} = \mathrm{diag}(1,-1,-1) \tag{1}$$

may be represented in terms of the Pauli matrices as $\gamma^0 = \sigma_3$, $\gamma^k = i\sigma_k$ or, equivalently, the matrices $(\alpha_1, \alpha_2) = \gamma^0(\gamma^1, \gamma^2) = (-\sigma_2, \sigma_1)$ and $\beta = \gamma^0$. Then the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form (we set $c = \hbar = 1$)

$$(i\partial_t - H_D)\Psi(t, \mathbf{r}) = 0, \tag{2}$$

where

$$H_D = \alpha \mathbf{P} + \beta m - eA^0 \equiv \sigma_1 P_2 - \sigma_2 P_1 + \sigma_3 m - eA^0 \tag{3}$$

is the Dirac Hamiltonian, $P_k = -i\partial_k + eA_k$ is the operator of generalized momentum of the electron, A_μ is the vector potential of the external electromagnetic field, m is the rest mass of the electron, and $-e$ ($e > 0$) is its electric charge. The Dirac wave function $\Psi(t, \mathbf{r})$ is a two-component function. In an external Coulomb field and a constant homogeneous magnetic field $B > 0$ along the z direction, the potential A_μ assumes the following forms in the symmetric gauge

$$A^0(r) = Ze/r \ (e > 0), \quad A_x = -By/2, \quad A_y = Bx/2. \tag{4}$$

We assume the wave functions to have the form

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{r}} \exp(-iEt)\psi_l(r, \varphi), \tag{5}$$

where E is the energy of the electron, and

$$\psi_l(r, \varphi) = \begin{pmatrix} F(r)e^{il\varphi} \\ G(r)e^{i(l+1)\varphi} \end{pmatrix} \tag{6}$$

with integral number l. The function $\psi_l(r, \varphi)$ is an eigenfunction of the conserved total angular momentum $J_z = L_z + S_z = -i\partial/\partial\varphi + \sigma_3/2$ with eigenvalue $j = l + \frac{1}{2}$. It should be reminded that l is not a good quantum number. Only the eigenvalues j of the conserved total angular momentum J_z are physically meaningful.

By putting Eqs. (5) and (6) into (2), and taking into account the equations

$$P_x \pm iP_y = -ie^{\pm i\varphi}\left(\frac{\partial}{\partial r} \pm \left(\frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{eBr}{2}\right)\right), \tag{7}$$

we obtain

$$\frac{dF}{dr} - \left(\frac{l+\frac{1}{2}}{r} + \frac{eBr}{2} \right) F + \left(E+m+\frac{Z\alpha}{r} \right) G = 0, \tag{8}$$

$$\frac{dG}{dr} + \left(\frac{l+\frac{1}{2}}{r} + \frac{eBr}{2} \right) G - \left(E-m+\frac{Z\alpha}{r} \right) F = 0, \tag{9}$$

where $\alpha \equiv e^2 = \frac{1}{137}$ is the fine structure constant. In a strong magnetic field the asymptotic solutions of $F(r)$ and $G(r)$ have the forms $\exp(-eBr^2/4)$ at large r, and r^γ with $\gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2}$ for small r. One must have $Z\alpha < \frac{1}{2}$, otherwise the wave function will oscillate as $r \to 0$ when $l=0$ and $l=-1$.

Let us assume

$$F(r) = r^\gamma \exp(-eBr^2/4)Q(r), \quad G(r) = r^\gamma \exp(-eBr^2/4)P(r). \tag{10}$$

In Ref. 13 we showed that parts of the spectrum could be analytically solved for by imposing the sufficient condition that $Q(r)$ and $P(r)$ be polynomials, thus showing that the system belongs to the QESM. The spectrum was solved in Ref. 13 from the recursion relations for the coefficients in the series expansion in Q and P. In this article, we will show that the same spectrum can also be obtained in a purely algebraic way. This is achieved by the method of factorization which leads to a set of Bethe ansatz equations.[13,14]

Substituting Eq. (10) into Eqs. (8) and (9) and eliminating $P(r)$ from the coupled equations, we have

$$\left\{ \frac{d^2}{dr^2} + \left[\frac{2\gamma}{r} - eBr + \frac{Z\alpha/r^2}{E+m+Z\alpha/r} \right] \frac{d}{dr} + E^2 - m^2 + \frac{2EZ\alpha}{r} + \frac{l+\frac{1}{2}}{r^2} - \frac{\gamma}{r^2} - eB(\Gamma+1) \right.$$

$$\left. + \frac{Z\alpha/r^2}{E+m+Z\alpha/r} \left[\frac{\gamma}{r} - eBr - \frac{l+\frac{1}{2}}{r} \right] \right\} \; Q(r) = 0, \tag{11}$$

where $\Gamma = l + \frac{1}{2} + \gamma$. Once $Q(r)$ is solved, the form of $P(r)$ is obtainable from Eqs. (8) and (10). If we let $x = r/l_B$, $l_B = 1/\sqrt{eB}$, Eq. (11) becomes

$$\left\{ \frac{d^2}{dx^2} + \left[\frac{2\gamma}{x} - x + \frac{Z\alpha}{x((E+m)l_Bx+Z\alpha)} \right] \frac{d}{dx} + (E^2-m^2)l_B^2 + \frac{2EZl_B\alpha}{x} + \frac{(l+\frac{1}{2}-\gamma)}{x^2} - (\Gamma+1) \right.$$

$$\left. - \frac{Z\alpha(l+\frac{1}{2}-\gamma)}{x^2[(E+m)l_Bx+Z\alpha]} - \frac{Z\alpha}{(E+m)l_Bx+Z\alpha} \right\} \; Q(x) = 0. \tag{12}$$

Equation (12) can be rewritten as

$$\left\{ \frac{d^2}{dx^2} + \left[\frac{2\beta}{x} - x - \frac{1}{x+x_0} \right] \frac{d}{dx} + \epsilon + \frac{b}{x} - \frac{c}{x+x_0} \right\} \; Q(x) = 0. \tag{13}$$

Here $\beta = \gamma + \frac{1}{2}$, $x_0 = Z\alpha/[(E+m)l_B]$, $\epsilon = (E^2-m^2)l_B^2 - (\Gamma+1)$, $b = b_0 + L/x_0$, $b_0 = 2EZ\alpha l_B$, $L = (l+\frac{1}{2}-\gamma)$, and $c = x_0 + L/x_0$. On expressing l_B in the expression of ϵ in terms of x_0, we get

$$\epsilon = \frac{E-m}{E+m} \left(\frac{Z\alpha}{x_0} \right)^2 - (\Gamma+1). \tag{14}$$

It is obvious that the energy E is determined once we know the values of ϵ and x_0. The corresponding value of the magnetic field B is then obtainable from the expression $l_B = Z\alpha/[(E+m)x_0]$. Solution of x_0 is achieved below by means of the Bethe ansatz equations.

III. THE BETHE ANSATZ EQUATIONS FOR $Q(x)$

We observe that the problem of finding the spectrum for Eq. (13) is equivalent to determining the eigenvalues of the operator

$$D = -\frac{d^2}{dx^2} - \left(\frac{2\beta}{x} - x - \frac{1}{x+x_0}\right)\frac{d}{dx} - \frac{b}{x} + \frac{c}{x+x_0}. \tag{15}$$

We want to factorize the operator (15) in the form

$$D = a^+ a + \epsilon. \tag{16}$$

The eigenfunctions of the operator D at $\epsilon = 0$ must satisfy the equation

$$aQ(x) = 0. \tag{17}$$

Suppose polynomial solutions exist for Eq. (13), say Q equals a nonvanishing constant, or $Q = \Pi_{k=1}^{n}(x - x_k)$, where x_k are the zeros of Q, and n is the degree of Q. In the case where Q is a constant (which may be viewed as corresponding to $n=0$), the operators a and a^+ have the form

$$a = \frac{d}{dx}, \quad a^+ = -\frac{d}{dx} - \left(\frac{2\beta}{x} - x - \frac{1}{x+x_0}\right). \tag{18}$$

If $Q = \Pi_{k=1}^{n}(x - x_k)$, a and a^+ will assume the form

$$a = \frac{d}{dx} - \sum_{k=1}^{n}\frac{1}{x - x_k} \tag{19}$$

and

$$a^+ = -\frac{d}{dx} - \left(\frac{2\beta}{x} - x - \frac{1}{x+x_0}\right) - \sum_{k=1}^{n}\frac{1}{x - x_k}. \tag{20}$$

We now substitute the forms of a and a^+ into Eq. (16) and compare the result with Eq. (15). This leads to conditions that must be satisfied by the various parameters and the roots x_k's. For constant Q ($n=0$), one has

$$\epsilon = b = c = 0. \tag{21}$$

The fact that $c=0$ implies

$$x_0^2 = -L. \tag{22}$$

For $n \geq 1$, one gets

$$b_0 + \frac{L}{x_0} = 2\beta\sum_{k=1}^{n}\frac{1}{x_k}, \quad \epsilon = n, \tag{23}$$

$$x_0 + \frac{L}{x_0} = \sum_{k=1}^{n}\frac{1}{x_k + x_0}, \tag{24}$$

$$\frac{2\beta}{x_k} - x_k - \frac{1}{x_k + x_0} - 2\sum_{j \neq k}^{n} \frac{1}{x_j - x_k} = 0, \quad k = 1, \ldots, n. \tag{25}$$

Equations (22), (24), and (25) constitute the set of $n+1$ Bethe ansatz equations relevant to this Dirac system, which involve $n+1$ unknown parameters $\{x_0, x_1, \ldots, x_n\}$. It is worthwhile to note that, unlike the corresponding equations in the Schrödinger and the Klein–Gordon cases discussed in Ref. 14, this set of Bethe ansatz equations involved not only the roots x_k, but also a parameter x_0 from D. Summing Eq. (25) over k leads to the expression $b_0 = x_0 + \Sigma_{k=1}^{n} x_k$, i.e., b_0 is simply the sum of all the roots of the Bethe ansatz equations. From the second equation in Eq. (23) we get

$$E^2 - m^2 = \frac{1}{l_B^2}(\Gamma + n + 1). \tag{26}$$

Since $-\frac{1}{2} \leq \Gamma \leq 0$ for $Z\alpha < \frac{1}{2}$,[13] we see from Eq. (26) that the solvable parts of the spectrum must satisfy $|E| \geq m$.

So we see that the solution of the solvable parts of the spectrum E boils down to solving the Bethe ansatz equations for x_0 in the differential operator, and the roots $x_k (k=1,\ldots,n)$ of $Q(x)$. Once the value of x_0 for each order $n = \epsilon$ is known, the energy E is given by Eq. (14). The corresponding magnetic field B is then determined from the definition of b_0, or from Eq. (26). The Bethe ansatz equations thus provide a systematic solutions of the QES spectrum. Of course, as the order of the degree of Q increases, analytical solutions of the Bethe ansatz equations becomes difficult, and one must resort to numerical methods.

IV. SOLUTIONS FOR $n=0,1$, and 2

In what follows we shall show the consistency of the solutions by the Bethe ansatz approach and that by the recursion relations presented in Ref. 13 for the first three lowest orders ($n = 0,1,2$) in Q. Instead of solving for x_0, our strategy is to eliminate it in Eq. (14) by means of Eqs. (22)–(25) so as to obtain an equation obeyed by E for each order of Q. This equation is then compared with the corresponding equation obtained from the recursion relations as presented in Ref. 13.

From Eqs. (21) and (22) we have $x_0^2 = -L$ and $\epsilon = 0$ when Q is a constant. Substitute these values of x_0 and ϵ into Eq. (14), and using the fact that $\Gamma L = (Z\alpha)^2$, we obtain the corresponding value of E as

$$E = -\frac{m}{2(l + \gamma + 1)}. \tag{27}$$

This is the result presented in Ref. 13. The corresponding allowed value of the magnetic field B is then obtained from Eqs. (26) and (27). The fact that x_0 is real leads to $L < 0$, which in turn implies that the energy levels given by Eq. (27) are only possible for $l < 0$. This is consistent with the conclusion obtained by the method of recursion relations.[13]

For $n = 1$, we find from Eqs. (14), (23), (24), and (25) that

$$\Gamma + 2 = \frac{E - m}{E + m}\frac{(Z\alpha)^2}{x_0^2}, \tag{28}$$

$$b_0 + \frac{L}{x_0} = \frac{2\beta}{x_1}, \tag{29}$$

$$\frac{1}{x_1 + x_0} = x_0 + \frac{L}{x_0}, \tag{30}$$

$$\frac{2\beta}{x_1} - x_1 - \frac{1}{x_1 + x_0} = 0. \tag{31}$$

Equations (29)–(31) imply $x_1 = b_0 - x_0$. Substituting x_1 into Eq. (30), we obtain

$$x_0^2 = L\left[\frac{E+m}{2E(Z\alpha)^2} - 1\right]^{-1}. \tag{32}$$

Then from Eqs. (32) and (28), we get

$$\left[4(\Gamma+1) - \frac{\Gamma}{Z^2\alpha^2}\right]E^2 + 4Em + \frac{\Gamma}{(Z\alpha)^2}m^2 = 0. \tag{33}$$

The energy E can be solved from Eq. (33) by the standard formula, after which the magnetic field is determined from Eq. (26). Equation (33) does not resemble the one obtained from the recursion relation in Ref. 13. However, on multiplying Eq. (33) by $\Gamma+1$ and making use of the fact that $(Z\alpha)^2 = \Gamma(\Gamma - 2\gamma)$, we can show, after some algebra, that Eq. (33) is equivalent to the corresponding equation given in Ref. 13.

   Finally we consider the case for $n=2$. We have Eq. (14) with $\epsilon=2$, together with Eqs. (23)–(25) in the forms

$$\Gamma + 3 = \frac{E-m}{E+m}\frac{(Z\alpha)^2}{x_0^2}, \tag{34}$$

$$b_0 + \frac{L}{x_0} = \frac{2\beta}{x_1} + \frac{2\beta}{x_2}, \tag{35}$$

$$\frac{1}{x_1 + x_0} + \frac{1}{x_2 + x_0} = x_0 + \frac{L}{x_0}, \tag{36}$$

$$\frac{2\beta}{x_1} - x_1 - \frac{1}{x_1 + x_0} - \frac{2}{x_2 - x_1} = 0, \tag{37}$$

$$\frac{2\beta}{x_2} - x_2 - \frac{1}{x_2 + x_0} - \frac{2}{x_1 - x_2} = 0. \tag{38}$$

From these equations we find $x_1 + x_2 = b_0 - x_0$ and $x_1 x_2 = 2\beta x_0(b_0 - x_0)/(b_0 x_0 + L)$. Putting these expressions into Eq. (36) and using the fact that $\Gamma = 2\beta + L - 1$, we arrive at

$$(b_0^2 - 2\beta)x_0^2 + b_0\Gamma x_0 + [b_0^2(L-1) - L(2\beta+1)] + \frac{b_0\Gamma L}{x_0} = 0. \tag{39}$$

Now multiplying Eq. (39) by Γ, using $\Gamma L = (Z\alpha)^2$, and expressing b_0, l_B, and $1/x_0^2$ in terms of E, we get finally

$$\left\{4(2\Gamma+3)-\frac{1}{(Z\alpha)^2}\left[6\Gamma+2(\gamma+1)+\frac{(2\gamma+1)\Gamma}{\Gamma+3}\right]\right\} E^3$$

$$+\left\{12-\frac{1}{(Z\alpha)^2}\left[2(\gamma+1)-\frac{(2\gamma+1)\Gamma}{\Gamma+3}\right]\right\} E^2 m$$

$$+\frac{1}{(Z\alpha)^2}\left[6\Gamma+2(\gamma+1)+\frac{(2\gamma+1)\Gamma}{\Gamma+3}\right]Em^2+\frac{1}{(Z\alpha)^2}\left[2(\gamma+1)-\frac{(2\gamma+1)\Gamma}{\Gamma+3}\right]m^3=0.$$

$$(40)$$

Again, this equation does not look the same as that obtained from the recursion relations. But we can show they are in fact equivalent as they differ only by a multiplicative factor $(\Gamma+1)(\Gamma+2)$.

V. THE BETHE ANSATZ EQUATIONS FOR $P(x)$

One may as well solve the QES energy spectrum of the problem from the differential equation of $P(x)$ instead of $Q(x)$. The analysis proceeds in exactly the same way as we did for $Q(x)$. We shall only give the outline below in order to show the similarity and differences between the two sets of Bethe ansatz equations.

The equation for $P(x)$ can be cast into the following form:

$$\left\{\frac{d^2}{dx^2}+\left[\frac{2\beta}{x}-x-\frac{1}{x+x_0'}\right]\frac{d}{dx}+\epsilon'+\frac{b'}{x}-\frac{c'}{x+x_0'}\right\} P(x)=0. \tag{41}$$

Here $x_0'=Z\alpha/[(E-m)l_B]$, $\epsilon'=(E^2-m^2)l_B^2-\Gamma$, $b'=b_0+c'$, and $c'=-\Gamma/x_0'$. Other parameters are as defined previously. Instead of Eq. (14) we now have

$$\epsilon'=\frac{E+m}{E-m}\left(\frac{Z\alpha}{x_0'}\right)^2-\Gamma. \tag{42}$$

We note here the sign difference before the mass terms in Eqs. (14) and (42). It is obvious that Eq. (41) is in the same form as Eq. (13), and hence is also quasi-exactly solvable. Suppose $P(x)$ has the factorized form $P(x)=\Pi_{k=1}^{n'}(x-x_k')$. Then the set of Bethe ansatz equations for the parameters $\{x_0',x_1',\ldots,x_{n'}'\}$ is given by

$$-\frac{\Gamma}{x_0'}=\sum_{k=1}^{n'}\frac{1}{x_k'+x_0'}, \tag{43}$$

$$\frac{2\beta}{x_k'}-x_k'-\frac{1}{x_k'+x_0'}-2\sum_{j\neq k}^{n'}\frac{1}{x_j'-x_k'}=0, \quad k=1,\ldots,n'. \tag{44}$$

In place of Eq. (23) we have

$$b_0-\frac{\Gamma}{x_0'}=2\beta\sum_{k=1}^{n'}\frac{1}{x_k'},$$

$$\epsilon'=n', \quad n'=1,2.\ldots \tag{45}$$

Summing Eq. (44) over k gives $b_0 = \Sigma_{k=1}^{n'} x_k'$. For any given integral value of $\epsilon' = n'$ the QES part of the energy E is determined from Eq. (42) once the values of x_0' are obtained from the Bethe ansatz equations. The corresponding value of the magnetic field B is then obtainable from the expression $l_B = Z\alpha/[(E-m)x_0']$.

We note here that since the two sets of Bethe ansatz Eqs. (22), (24) and (25), and (43) and (44) give the same spectrum of the QES energy E and the corresponding B, we have, from the values of ϵ, ϵ' and b_0, the following necessary conditions:

$$n' = n + 1, \tag{46}$$

$$b_0 = x_0 + \sum_{k=1}^{n} x_k = \sum_{k=1}^{n+1} x_k'. \tag{47}$$

Conversely, one can easily show that Eqs. (46) and (47) are also the sufficient conditions for the two sets of Bethe ansatz equations to give the same QES energy spectrum and magnetic field. The condition (46) implies that the degree of the polynomial $P(x)$ is of one order higher than that of $Q(x)$, which is in complete agreement with the result obtained in Ref. 13.

VI. NON-sl_2-BASED QUASI-EXACTLY SOLVABILITY

We now demonstrate that the QES Eqs. (13) and (41) cannot be represented as a bilinear combination of the generators of the sl_2 algebra. The question of whether there exists non-sl_2-based one-dimensional QESM was first posed in Ref. 2 in which all sl_2-based QESM are classified. The first example of such a kind was given in Ref. 16, which presents a potential arising in the context of the stability analysis around the kink solution for ϕ^4-type field theory in $1+1$ dimensions.

We shall show that Eq. (13) is not generated by the sl_2 algebra. The same conclusion applies immediately to Eq. (41), since both equations have the same form. Let us rewrite Eq. (13) as

$$\left\{ -(x^2 + x_0 x)\frac{d^2}{dx^2} + [x^3 + x_0 x^2 + (1-2\beta)x - 2\beta x_0]\frac{d}{dx} - \epsilon x^2 + (c - b - \epsilon x_0)x - bx_0 \right\} Q(x) = 0. \tag{48}$$

Turbiner[2] has shown that all sl_2-based second order QES differential equations can be cast into the form

$$-P_4(x)\frac{d^2 Q}{dx^2} + P_3(x)\frac{dQ}{dx} + (P_2(x) - \lambda)Q = 0, \tag{49}$$

where

$$P_4(x) = a_{++}x^4 + a_{+0}x^3 + (a_{+-} + a_{00})x^2 + a_{0-}x + a_{--},$$

$$P_3(x) = 2(2j-1)a_{++}x^3 + [(3j-1)a_{+0} + b_+]x^2 + [2j(a_{+-} + a_{00}) + a_{00} + b_0]x + ja_{0-} + b_-, \tag{50}$$

$$P_2(x) = 2j(2j-1)a_{++}x^2 + 2j(ja_{+0} + b_+)x + a_{00}j^2 + b_0 j.$$

Here a_{kl}'s and b_k's ($k,l = +,0,-$) are constants, and j is a non-negative integer or half-integer. Equation (49) corresponds to the eigenvalue problem

$$HQ = \lambda Q, \quad H = -\sum_{\substack{k,l=+,0,- \\ k \geq l}} a_{kl} J^k J^l + \sum_{k=+,0,-} b_k J^k, \tag{51}$$

which has a polynomial solution of power $2j$ in x. Here J^k's are the generators of sl_2:

$$J^+ = x^2\frac{d}{dx} - 2jx, \quad J^0 = x\frac{d}{dx} - j, \quad J^- = \frac{d}{dx}. \tag{52}$$

Comparing Eqs. (48) and (49) we find that the two equations are inconsistent with each other. For instance, the coefficient of x^4 in P_4 requires $a_{++} = 0$, whereas the coefficient of x^3 in P_3 implies $2(2j-1)a_{++} = 1$, which gives a nonvanishing a_{++} for positive integral and half-integral values of j. This shows that Eq. (13) is not sl_2-based.

VII. CONCLUSIONS

In conclusion, we have given an algebraic solution to the planar Dirac equation of an electron in the Coulomb and magnetic fields. The relevant Bethe ansatz equations are presented. Unlike the corresponding equations in the Schrödinger and the Klein-Gordon case discussed in Ref. 14, the unknown variables in this set of Bethe ansatz equations include not only the roots of the polynomial assumed, but also a parameter from the QES differential operator. Equivalence between this approach and that by the recursion relations is demonstrated. Finally, we show that the QES equation for this problem does not belong to any of the classes based on the sl_2 algebra.

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