A Clifford algebra quantization of Dirac's electron positron field

H. T. Cho, Adel Diek, and R. Kantowski
Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019

(Received 23 June 1989; accepted for publication 9 May 1990)

The quantum field theory of free Dirac particles (four-component massive spin-$\frac{1}{2}$ particles) is "derived" by a Segal quantization procedure. First, details are given on how the spinor space of Dirac is actually a minimal left ideal of the Clifford algebra associated with a Lorentz inner product space $(+,-,-,-)$, and how the homogeneous group actions break the natural two-component quaternion structure to give familiar four-component complex spinors. Second, Wigner's procedure for constructing unitary representations of the Poincaré group is used to construct the appropriately induced infinite-dimensional representation of the inhomogeneous group starting from the above four-dimensional nonunitary representation. Third, and finally, Segal's procedure for quantizing classical Fermion fields is adapted to this infinite-dimensional Hilbert space to obtain the Clifford algebra of annihilation–creation operators for spin-$\frac{1}{2}$ particles. The familiar Fock space appears as a minimal left ideal in this second Clifford algebra.

I. INTRODUCTION

In this paper we show how the familiar quantum field theory of free massive Dirac spin-$\frac{1}{2}$ particles[1,2] can be obtained by two successive Clifford algebra constructions. We refer to this generic field as the electron–positron field for convenience, and have attempted to use notation familiar to the physics community when possible.

In Sec. II we give a short introduction to the standard construction of the Clifford algebra associated with a real vector space possessing a nondegenerate inner product.[3,4] This construction is applied to an infinite-dimensional space in Sec. IV. However, in Sec. II emphasis is placed on a Minkowski inner-product space with signature -2 and its corresponding Clifford algebra of gamma matrices. The technique for generating spinor representations of the associated Clifford algebras is given and applied to the four-dimensional case.[5–8] By carefully including the discrete transformations (parity and time reversal) we are able to show how the presence of projective representations and additional group actions (i.e., phase rotation and charge conjugation) in the Dirac theory destroys the expected two-component quaternion structure of spinors for the $(+,-,-,-)$ metric. We assume the homogeneous group structure to consist of a covering group of the homogeneous Lorentz group and the above additional members. Throughout we use the four-dimensional spinor basis corresponding to rest states having spins oriented along the $\pm z$ axis and possessing ± 1 parity. Our motive is to use an explicit basis that produces the Pauli–Dirac representation of the gamma matrices familiar to all physicists.[1,2] Other representations such as Weyl or Majorana could easily be used.

In Sec. III we use Wigner's procedure for constructing unitary representations of the Poincaré group to construct a representation of the inhomogeneous group obtained by combining the above homogeneous group with Minkowski space translations.[9–14] Two ingredients are critical and both make use of the four-component Dirac representation of Sec. II. The first is a character, or equivalently a one-dimensional unitary representation of the translations (massive for the case considered here), and the second is a unitary representation of the Little group of this character (the invariance group of this character). Both are found in the above four-component representation; e.g., when the homogeneous group is restricted to the Little subgroup, the spin representation becomes unitary.

The reason for constructing this representation and its infinite-dimensional Hilbert space is that in Sec. IV a second Clifford algebra, the operator algebra of the electron–positron theory, is constructed. By a straightforward extension of the general construction outlined in Sec. II, and previously attributed to Segal,[15–21] this complex Clifford algebra is constructed from the infinite-dimensional complex Hilbert space. All the general notions introduced in Sec. II can be applied to this infinite-dimensional Clifford algebra. In particular, a projection operator (the Fock vacuum) is used to generate the space of spinors (the Fock space).[22] This is a new construction differing from a previously introduced Fock space.[18] Prior to Sec. IV the only complex structure present came from the four-component spinor representation of the first Clifford algebra and appeared in the unitary representation of Sec. III; however, in Sec. IV another complex structure in the second Clifford algebra appears.

In this paper we have tried to "draw" a straight line from Minkowski space to the quantum field Ψ, however, as the reader will obviously notice we made several choices (usually among a few possibilities) along the way. Since the Dirac theory is the standard theory for electrons and positrons, we have used it as a guide to make the appropriate choices.

II. THE MINKOWSKI CLIFFORD ALGEBRA AND DIRAC SPINORS

This section serves primarily to establish needed background and notation. However, inclusion of group actions, beyond special Lorentz, is new and allows us to clarify why Dirac spinors are four-component complex and not two-component quaternion when the signature is -2.[3–8,23–25]

The 16-dimensional real Clifford algebra $R_{1,3}$ can be attached to each point $x\in M^4$ of Minkowski space by selecting a translationally invariant set of basis vectors $e_\mu \cong \partial_\mu$ satisfying $e_\mu \cdot e_\nu = g_{\mu\nu} = \text{diagonal}(1,-1,-1,-1)$ to span the tangent space at each x. In general, the universal Clifford algebra (CA) associated with a real vector space V possessing a nondegenerate symmetric bilinear form (,) is the unique associative algebra:

(i) with identity I,

(ii) generated by a subspace $V^1 \subset \text{CA}$ isomorphic to V,

(iii) which has its algebraic multiplication constrained by

$$v^1 w^1 + w^1 v^1 = 2(v,w)I. \tag{2.1}$$

For M^4 we write $v = r^\mu e_\mu$, and the isomorphic vector images in $V^1 \subset \text{CA}$ as $v^1 = r^\mu \gamma_\mu$. The defining algebraic constraint (iii) familiarly appears as $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}I$.

Using orthonormal frames such as these $(e_\mu \leftrightarrow \gamma_\mu)$ allows the 16-dimensional real Clifford algebra $R_{1,3}$ to be decomposed into a direct sum of Lorentz scalars, vectors, bivectors, pseudovectors, and pseudoscalars $R_{1,3} = V^0 \oplus V^1 \oplus V^2 \oplus V^3 \oplus V^4$, where

$$\begin{aligned} &V^0 = \{rI\}, \quad V^1 = \{r^\mu \gamma_\mu\}, \quad V^2 = \{r^{\mu\nu}\gamma_\mu\gamma_\nu\},\\ &V^3 = \{r^{\mu\nu\lambda}\gamma_\mu\gamma_\nu\gamma_\gamma\} = \{r^\mu\gamma_\mu\gamma\},\\ &V^4 = \{r\gamma\} \quad \text{where } \gamma \equiv \gamma_0\gamma_1\gamma_2\gamma_3, \\ &\text{with } r, r^\mu, r^{\mu\nu}, \text{ and } r^{\mu\nu\lambda} \text{ real},\\ &\mu<\nu<\lambda, \quad \text{and } \gamma\gamma = -I. \end{aligned} \tag{2.2}$$

In general, the even subalgebra $\text{CA}^+ \equiv \{V^0 \oplus V^2 \oplus V^4 \oplus \cdots\}$ of a Clifford algebra is isomorphic to another Clifford algebra. In the case of $R_{1,3}$ the even subalgebra is isomorphic to $R_{3,0}$ the eight-dimensional real algebra associated with the three-dimensional Euclidean space. The algebra $R_{3,0}$ is isomorphic to the four-dimensional complex Pauli algebra because the center of $R_{3,0}$ is isomorphic to the complex numbers. Then,

$$\begin{aligned} R^+_{1,3} &= \{rI\} \oplus \{r^{\mu\nu}\gamma_\mu\gamma_\nu\} \oplus \{r\gamma\}\\ &= \{rI\} \oplus \{r^i\alpha_i\} \oplus \{r^i\alpha_i\alpha\} \oplus \{r\alpha\}\\ &\cong R_{3,0} = \{rI\} \oplus \{r^i\sigma_i\} \oplus \{r^i\sigma_i\sigma\} \oplus \{r\sigma\}\\ &\cong \text{Pauli} = \{(r+ir')\} \oplus \{(r^i + ir'^i)\sigma_i\}, \end{aligned} \tag{2.3}$$

where $\alpha_i \equiv \gamma_i\gamma_0$, $\alpha \equiv \alpha_1\alpha_2\alpha_3 = \gamma$. The center of $R_{3,0}$ is $\{rI\} \oplus \{r'\sigma\} \cong \{(r+ir')I\}$. In (2.3) the σ_i are images of an orthonormal basis ϵ_i, $(\epsilon_i \cdot \epsilon_j = \delta_{ij})$ of a three-dimensional Euclidean space and generate the associated Clifford algebra $R_{3,0}$. The isomorphism to the familiar complex Pauli matrix form is $\sigma_i \leftrightarrow$Pauli matrix σ_i, $\sigma \leftrightarrow$imaginary unit "i." The precise identification of $R^+_{1,3}$ with $R_{3,0}$ requires a choice of observer e_0 and its corresponding γ_0 in V^1. By identifying $\sigma_i \leftrightarrow \alpha_i = \gamma_i\gamma_0$, then $i \leftrightarrow \alpha = \gamma$ and we have the desired even-subalgebra isomorphism. We also have the decomposition of the total Clifford algebra,

$$R_{1,3} \cong R_{3,0} \oplus \gamma_0 R_{3,0}, \tag{2.4}$$

allowing (anti)automorphisms of $R_{3,0}$ to be extended to $R_{1,3}$ by defining their effect on γ_0. Every universal Clifford algebra associated with a real or complex vector space possesses a fundamental antiautomorphic involution called reversion $c \to \tilde{c}$ and a fundamental automorphic involution called inversion $c \to \bar{c}$. The defining properties are

$$\begin{aligned} &\widetilde{c_1c_2} = \tilde{c}_2\tilde{c}_1, \quad \tilde{c} = c, \quad c \in V^1,\\ &\overline{c_1c_2} = \bar{c}_1\bar{c}_2, \quad \bar{c} = -c, \quad c \in V^1. \end{aligned} \tag{2.5}$$

Vectors are invariant under reversion but are reversed in direction by inversion. For $R_{1,3}$ we denote reversion by tilde as above but for inversion we can write

$$\bar{c} = \gamma_5(c) \equiv \gamma c \gamma^{-1}, \tag{2.6}$$

where we have used the notation γ_5 of the Dirac helicity operator. For $R_{3,0}$ we use $c \to c^\ddagger$, the familiar Hermitian conjugation for the Pauli algebra, for which $\alpha_i^\ddagger = \alpha_i$, $\alpha^\ddagger = (\alpha_1\alpha_2\alpha_3)^\ddagger = \alpha_3\alpha_2\alpha_1 = -\alpha$. Equation (2.4) allows Pauli reversion to be extended to $R_{1,3}$ by requiring $\gamma_0^\ddagger = \gamma_0$ where

$$c \to c^\ddagger \equiv \gamma_0\tilde{c}\gamma_0^{-1} \Rightarrow \gamma_i^\ddagger = -\gamma_i. \tag{2.7}$$

The Pin group is a subgroup of the multiplicative group of invertible elements in CA that leave the subspace V^1 invariant when acting as inner automorphisms, i.e.,

$$c \in \text{Pin} \Leftrightarrow cV^1c^{-1} = V^1, \quad \text{and satisfy } \tilde{c}c = \pm I. \tag{2.8}$$

With a choice of observer γ_0 in $R_{1,3}$, the $\text{Pin}_{1,3}$ constraint (2.8) can be written

$$c^\ddagger\gamma_0 c = \pm\gamma_0. \tag{2.9}$$

The subgroup connected to the identity is isomorphic to SL(2,C) and is generated by products of rotations and boosts,

$$\begin{aligned} &\text{Rotations} = e^{(\theta^i/2)\gamma\alpha_i}, \quad \sqrt{\theta^i\theta^i} \leqslant 4\pi,\\ &\text{Boosts} = e^{(\zeta^i/2)\alpha_i}, \quad -\infty < \zeta^i < \infty,\\ &\text{SL(2,C)} \cong \left\{e^{[(\theta^i/2)\gamma + \zeta^i/2]\alpha_i}\right\}. \end{aligned} \tag{2.10}$$

The three other disconnected parts of $\text{Pin}_{1,3}$ are generated by products of the identity component with parity P (multiplication by γ_0) and time reversal T (multiplication by $\gamma_1\gamma_2\gamma_3$). The identity and parity components satisfy $\tilde{c}c = +I$ or equivalently $c^\ddagger\gamma_0 c = +\gamma_0$ whereas the T and PT components satisfy $\tilde{c}c = -I$ or equivalently $c^\ddagger\gamma_0 c = -\gamma_0$. When Pin acts as inner automorphisms on V^1 (called the vector representation) it double covers the invariance group of V's inner product (for $R_{1,3}$ the invariance group is the homogeneous Lorentz group),

$$\gamma_\mu \to (\pm c)\gamma_\mu(\pm c)^{-1} = \gamma_\nu\Lambda^\nu{}_\mu, \tag{2.11}$$

where $\Lambda^\nu{}_\mu$ is a Lorentz matrix. The spin representation of Pin arises by letting Pin act as left multiplications on CA. It is reducible with each invariant subspace giving a spinor space. A CA is decomposed into a direct sum of minimal left ideals, called spinor spaces CA_n, by finding a complete set of mutually annihilating primitive idempotents (projection operators) P_n,[26]

$$P_nP_m = \delta_{n,m}P_n, \quad I = \sum_n P_n \Rightarrow \text{CA} = \sum_n \text{CA}_n, \quad \text{where } \text{CA}_n \equiv \text{CA}\,P_n. \tag{2.12}$$

Decomposition of $R_{1,3}$ requires two idempotents. To obtain matrix representations of the γ_μ's familiar to the physics community, we choose an observer (γ_0) and construct a pair of projection operators $P_\pm$ using a unit spatial direction α_3 ($\alpha_3^2 = I$) in the even subalgebra,

$$P_\pm \equiv \tfrac{1}{2}(I \pm \alpha_3), \Rightarrow R_{1,3} = R_{1,3+} \oplus R_{1,3-}. \qquad (2.13)$$

It is at this point that clear differences in the fields of $R_{1,3}$, $R_{1,3\pm}$, and the complex numbers required for Dirac theory begin to appear. Both spinor spaces $R_{1,3\pm}$ as subspaces of $R_{1,3}$ form eight-dimensional vector spaces over the reals, but if they are used only as representation spaces for left multiplication by $\text{Pin}_{1,3}$, they form two-dimensional vector spaces over the field of quaternions. However, Dirac theory contains an additional continuous U_1 group action (a right multiplication on $R_{1,3}$), uses a projective representative for time reversal T rather than using left multiplication by $\mathsf{T} = \gamma_1\gamma_2\gamma_3$, and introduces a projective representative for the additional charge conjugation symmetry C, all of which are inconsistent with the quaternion structure of $R_{1,3\pm}$. These new group actions "break" the two-component quaternion structure leaving a four-component complex structure for each spinor space. The remaining complex structure is defined by right multiplication by γ, i.e., multiplication by the unit imaginary "i" of the complex field is defined by $i(c) \equiv c\gamma$. Using the above minimal left ideals ($R_{1,3\pm}$) the U_1 group action on $R_{1,3}$ can be taken as right multiplication by

$$U_1(\phi) = e^{\phi\gamma\alpha_3} \quad (0 \leqslant \phi < 2\pi), \qquad (2.14)$$

rotating the phases of the two spinor spaces $R_{1,3\pm}$ oppositely. The quaternion structure "broken" by (2.14) but not by left multiplications (spin transformations) is generated by right multiplications by γ_1, γ_2, and $\gamma_1\gamma_2$. These commute with the projection operators $P_\pm$ leaving $R_{1,3\pm}$ invariant and obviously commute with left multiplications. The full 16-dimensional real Clifford algebra could be represented by 2×2 quaternion matrices using for example w_1 and w_3 from (2.15) below as basis vectors of $R_{1,3+}$.[23,24,27] Because (2.14) does not commute with γ_1 or γ_2, two additional basis vectors must be introduced to represent the U_1 needed in Dirac theory. To obtain the familiar Pauli–Dirac matrix representation for the γ_μ's, we use the following w_Ω basis of $R_{1,3+}$ and to obtain the associated z axis oriented, positive and negative energy spinors $u_{A,\mathbf{p}}$ and $v_{A,\mathbf{p}}$, we use the associated basis z_Ω:

$$\begin{aligned} &w_1 \equiv (I+\gamma_0)P_+, \quad z_1 \equiv (I+\gamma_0)P_+, \\ &w_2 \equiv (I+\gamma_0)\alpha_1 P_+, \quad z_2 \equiv (I+\gamma_0)\alpha_1 P_+, \\ &w_3 \equiv (I-\gamma_0)P_+, \quad z_3 \equiv (I-\gamma_0)\alpha_1 P_+, \\ &w_4 \equiv (I-\gamma_0)\alpha_1 P_+, \quad z_4 \equiv (I-\gamma_0)P_+. \end{aligned} \qquad (2.15)$$

Using

$$\gamma_\mu w_\Omega \equiv w_\Lambda \Gamma_{\mu\Omega}^{\ \Lambda}, \quad \text{where } (\Omega,\Lambda) = \{1,2,3,4\} \qquad (2.16)$$

gives

$$\begin{aligned} &\gamma_0 \cong \Gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_i \cong \Gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \\ &\gamma \cong \Gamma \equiv \Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, \end{aligned} \qquad (2.17)$$

and

$$\gamma_5(w_\Omega) = \gamma w_\Omega \gamma^{-1} \equiv w_\Lambda \Gamma_{5\Omega}^{\ \Lambda}, \Rightarrow \Gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where in (2.17) I is the 2×2 identity, σ_i are the standard Pauli matrices, and γ on the right (defining the complex structure) has been replaced by multiplication of the imaginary unit "i" from the complex field. If we were interested in the Dirac wave theory we could and would now go to the above matrix representation; however, since we are interested in the Dirac quantum field theory, which requires a different complex structure (see Sec. IV), we keep the concrete basis picture w_Ω with right multiplication by γ. It should be observed that the representation matrices for the Clifford algebra (2.16) and (2.17) are unaffected by a change of basis $w_\Omega \to w_\Omega e^{\phi\gamma\alpha_3} = w_\Omega e^{\phi\gamma}$. This invariance constitutes the global U_1 invariance of the free electron–positron theory and it along with the following $\star$ innerautomorphism of $R_{1,3}$ are at the core of the projective representatives of time reversal and charge conjugation actions on spinors. Given a basis for $R_{1,3+}$ such as w_Ω, and its complex structure mapping γ, Pauli reversion ‡ of (2.7) can be decomposed into a commuting pair of involutions, $\star$ and ^{T}, called complex conjugation and transposition,

$$(\overset{\star}{c})^T = (c^T)^\star = c^\ddagger, \quad (c^T)^T = c, \quad (\overset{\star}{c})^\star = c,$$

constrained by

$$\overset{\star}{w}_\Omega = w_\Omega, \quad \overset{\star}{\gamma} = -\gamma, \quad \Rightarrow \gamma^T = \gamma. \qquad (2.18)$$

Complex conjugation $\star$ is a pure innerautomorphism, and transposition is $\star$ followed by ‡. They are defined in terms of an element $C \in R_{1,3}$ that depends on the spinor basis,

$$\begin{aligned} (\overset{\star}{c}) &= (C\gamma_0\gamma)c(C\gamma_0\gamma)^{-1} = (C\gamma_0)\gamma_5(c)(C\gamma_0)^{-1} \\ &= C\tilde{\bar{c}}^\ddagger C^{-1}, \end{aligned}$$

$$c^T = \overset{\star}{c}{}^\ddagger = C\tilde{\bar{c}}\, C^{-1}, \Rightarrow C\gamma_\mu C^{-1} = -\gamma_\mu^T,$$

where

$$C^\ddagger = C^{-1}, \quad \tilde{\bar{C}} = \pm C. \qquad (2.19)$$

For the basis (2.15) we have

$$C = \pm\alpha_2 \Rightarrow \overset{\star}{\gamma}_0 = \gamma_0, \quad \overset{\star}{\gamma}_1 = \gamma_1, \quad \overset{\star}{\gamma}_3 = \gamma_3,$$

and

$$\overset{\star}{\gamma}_2 = -\gamma_2. \qquad (2.20)$$

The constraints of (2.18) were placed on $\star$ so that the representative matrices (2.17) also satisfy (2.20) where $\star$ becomes complex conjugation of components.

The discrete spinor transformations of Dirac theory representing parity P, time reversal T, and charge conjugation C (written in capital bold Roman letters) are:

$$\begin{aligned} &\mathbf{P}(c) \equiv \pm\gamma_0 c, \\ &\mathbf{T}(c) \equiv \tilde{c}^\ddagger C^{-1} e^{\phi_T\gamma\alpha_3} = -\gamma_0 c\gamma_2 e^{\phi_T\gamma\alpha_3} \quad (C = +\alpha_2), \\ &\mathbf{C}(c) \equiv \gamma c\gamma_0 C^{-1} e^{\phi_C\gamma\alpha_3} = -\gamma c\gamma_2 e^{\phi_C\gamma\alpha_3}, \end{aligned} \qquad (2.21)$$

where ϕ_T and ϕ_C are arbitrary real constants. Of these three, only the parity representative could be guessed without prior knowledge of the Dirac wave theory. The choice of "$\pm$" is left to convention, both generate the same connected part of the group and both have the same effect on vectors. The projective representative of time reversal is antilinear, satisfies $\mathbf{T}^2 = -I$, does not change parity or charge, but flips the spin eigenvalue. The projective spin representative of charge conjugation is antilinear, satisfies $\mathbf{C}^2 = I$, flips the parity and the charge, but not the spin. We give the matrix representatives of the discrete transformations using the z_Ω basis (2.15) rather than the w_Ω basis because we need them in Sec. III:

$$\mathbf{P}(z_\Omega) \equiv z_\Lambda \mathbf{P}^\Lambda_\Omega, \quad \mathbf{T}(z_\Omega) \equiv z_\Lambda \mathbf{T}^\Lambda_\Omega, \quad \mathbf{C}(z_\Omega) \equiv z_\Lambda \mathbf{C}^\Lambda_\Omega,$$

where

$$\mathbf{P} = \pm\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}. \tag{2.22}$$

In the above we have described a representation of a group we call the homogeneous group H without discussing the structure of H itself. Our point of view is that H consists of eight disconnected parts, the identity component being $\mathrm{Sl}(2,\mathrm{C}) \times \mathrm{U}_1$ and the other seven given by products of $\mathrm{Sl}(2,\mathrm{C}) \times \mathrm{U}_1$ and one or more of P, T, and C. The structure of this homogeneous electron–positron invariance group is a direct product of $\mathrm{Pin}_{1,3}$ and the gauge group G, which consists of the phase rotations U_1 and charge conjugaton C,

$$H = \mathrm{Pin}_{1,3} \times G. \tag{2.23}$$

The $\mathrm{Pin}_{1,3}$ group consists of the four disconnected parts described below (2.10) and the $G \cong \mathrm{U}_1 \circledS \{I,\mathrm{C}\}$ group consists of two parts, giving eight all together. In G the semidirect product action of C on U_1 is $u \to u^{-1}$. The spin representatives of H are just the transformations of $R_{1,3}$ generated by products of left multiplication by (2.10), right multiplication by (2.14), and (2.22) actions; and leave invariant a Hermitian inner product on $R_{1,3}$ (considered as an eight-dimensional complex representation space with γ on the right defining the complex structure). The Hermitian (Dirac) inner product is the familiar one constructed using Pauli reversion,

$$\langle c_1,c_2 \rangle \equiv (c_1^\ddagger \gamma_0 c_2)_{s+ps} = (c_2^\ddagger \gamma_0 c_1)^\ddagger_{s+ps} = \langle c_2,c_1 \rangle^\ddagger, \tag{2.24}$$

where $s+ps$ stands for scalar and pseudoscalar (i.e., $V^0 \oplus V^4$) parts only. When † is applied to the unit pseudoscalar γ in (2.2) it changes its sign, i.e., changes i to $-i$ as required of Hermitian inner products. In other words, (2.24) says that to compute the Hermitian inner product of c_1 and c_2 considered as two eight-dimensional complex vectors in $R_{1,3}$, use the 16-dimensional real Clifford algebra multiplication to evaluate $c_1^\ddagger \gamma_0 c_2$ and keep only the scalar and pseudoscalar parts, remembering that the unit pseudoscalar γ of the Clifford algebra, when acting on the right, is equivalent to multiplying by the unit imaginary "i" of the complex field. We observe that because $\langle R_{1,3+}, R_{1,3-} \rangle = 0$, the Hermitian inner product provides an inner product on each spinor subspace separately, i.e., on $R_{1,3+}$; it is the familiar Dirac $\overline{\Psi}\Phi$,

$$\langle w_\Omega, w_\Lambda \rangle = \langle z_\Omega, z_\Lambda \rangle = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.25}$$

where z_Ω and w_Ω are basis vectors of (2.15). We also observe that when the Boosts are excluded from the homogeneous group, invariance of (2.24) is equivalent to invariance of the Hilbert space inner product

$$\langle\langle c_1,c_2 \rangle\rangle \equiv (c_1^\ddagger c_2)_{s+ps} = (c_2^\ddagger c_1)^\ddagger_{s+ps} = \langle\langle c_2,c_1 \rangle\rangle^\ddagger, \tag{2.26}$$

which on $R_{1,3+}$ is simply

$$\langle\langle z_\Omega, z_\Lambda \rangle\rangle = \delta_{\Omega\Sigma}. \tag{2.27}$$

With respect to this inner product, the identity and parity component representatives are unitary whereas the time reversal and charge conjugation component representatives are projective and antiunitary.

For a global picture we wish to look at the homogeneous group H as the Lie group of a trivial principal fiber bundle HB over M^4,

$$HB \cong M^4 \times H \to M^4. \tag{2.28}$$

We call this the homogeneous bundle and think of it as an enlargement of the bundle of orthonormal frames for M^4 (whose global gauge group is the homogeneous Lorentz group and whose fibers consist of only four disconnected parts). We take this point of view primarily to avoid double-valued representations. It is sometimes beneficial (but incorrect) to think of HB as the spinor frame bundle. One obvious incorrectness occurs because of the projective action of the T and C components on the spinor frames. The invariance group P, of Dirac's free electron–positron theory is a semidirect product of space-time translations R^4 and the homogeneous group H,

$$P = R^4 \circledS H, \tag{2.29}$$

and can be thought of as the Poincare group enlarged to remove double-valued representations of the Lorentz group as well as to include U_1 and charge conjugation C. In (2.29), H is the isotropy of the origin $(x=0)$. The inhomogeneous group acts as fiber preserving mappings on HB, $(r,h) \in P$ acts on $(x,h') \in HB$ by

$$(x,h') \to (r + h(x), hh'), \tag{2.30}$$

where the h action on x is the expected vector action for $h \in \mathrm{Pin}_{1,3}$ and is ineffectual for $h \in G$. Here, P is seen to act as a group of bundle automorphisms because its action commutes with the H action of HB. The isotropy subgroup $H_x \subset P$ is isomorphic to H and consists of those inhomogeneous transformations that leave $x \in M^4$ invariant,

$$H_x \equiv \{(x - h(x), h) \in P\} \cong H. \tag{2.31}$$

III. WIGNER'S INDUCED REPRESENTATION PROCEDURE

Dirac electron–positron theory contains another representation of the semidirect product (2.29) beyond the four-component spin representation given in Sec. II. In this section we use Wigner's procedure for constructing induced representations to construct this second representation that is in fact faithful, irreducible, and unitary[9–14] We are necessarily careful to keep track of the gauge group action and the discrete transformations. In Sec. IV we apply Segal's quantization procedure to this infinite-dimensional representation and arrive at free positron–electron quantum field theory.

To construct a unitary representation of the inhomogeneous group $R^4 \circledS H$, Wigner's procedure requires first the selection of a one-dimensional representation of the translations (often called a character);

$$\chi: R^4 \to U_1 \Leftrightarrow \chi(r) = e^{-\gamma p_\mu r^\mu} = e^{-\gamma mc r^0}, \tag{3.1}$$

to which we have already adapted a global frame e_μ (fixes e_0 only). We have selected a character appropriate for a massive particle and, by using the complex structure mapping γ rather than imaginary field unit "i," indicated that the translations will act on the right of the four-component spinor space $R_{1,3+}$. The subgroup $L \subset H$ whose vector action leaves e_0 (and hence χ) invariant is called the Little group and is generated by products of spatial rotations $SU_2 \subset SL(2,C)$, U_1 phase rotations, parity P, and charge conjugation C. The Little group L thus consists of four disconnected parts: the indentity component $SU_2 \times U_1$ and three other components generated by multiplications with P and C. The second needed ingredient in the Wigner construction is an irriducible unitary representation of L. In the electron–positron case this representation is given by the $L \subset H$ actions on $R_{1,3+}$ described in Sec. II. Even though, as a representatin of H, the four-component spin representation is not unitary, as a representation of the subgroup L, it is [see (2.26)]. Under L actions, $R_{1,3+}$ decomposes into the direct sum of two orthogonal two-component Pauli spinor subspaces of opposite parity,

$$\begin{aligned} R_{1,3+} &= {}_+R_{1,3} \oplus {}_-R_{1,3+}, \\ {}_\pm R_{1,3+} &\equiv [(I \pm \gamma_0)/2] R_{1,3+}. \end{aligned} \tag{3.2}$$

Each is invariant under SU_2 and parity P [**P** of (2.21)] but is exchanged by the action of charge conjugation C [**C** of (2.21)]. This is easily seen by choosing (z_1,z_2) and (z_3,z_4) of (2.15) as respective pairs of basis vectors.

The next step in the Wigner construction is to define the infinite-dimensional complex vector space $\mathscr{H}$ of functions from $H/L \cong$ Boosts (part connected to coset L only) into the representation space for L, i.e., into $R_{1,3+}$. Time reversal actions are defined on these functions. The Boost actions on translations R^4 pull back to actions on the set of characters and make the Boosts topologically equivalent to the upper mass shell. In particular,

$$e^{(\zeta^i/2)\alpha_i} \to e^{-\gamma mc \Lambda^0_\mu r^\mu} = e^{-\gamma p_\mu r^\mu}, \tag{3.3}$$

where the Boost parameters ζ^i are related to the mass shell point by (2.11),

$$e^{(\zeta^i/2)\alpha_i} \gamma_0 e^{-(\zeta^i/2)\alpha_i} = \gamma_\mu \Lambda^\mu_0 = \gamma_\mu p^\mu / mc. \tag{3.4}$$

Consequently, $\mathscr{H}$ is equivalent to functions from the upper mass shell to $R_{1,3+}$, i.e., four-component spinor valued functions of p^μ, $\psi(p^\mu) = z_\Omega \psi^\Omega(p^\mu)$. Notice that the components $\psi^\Omega(p^\mu)$ appear on the right of the basis vectors z_Ω because the complex structure has been defined as right multiplication by γ.

The induced action $U_{(r,h)}$ of $(r,h) \in R^4 \circledS H$ on $\psi \in \mathscr{H}$ is, for $s \in SL(2,C) \subset H$,

$$[U_{(0,s)}\psi](p^\nu) = e^{-(\zeta^i/2)\alpha_i} S e^{(\zeta'^i/2)\alpha_i} \psi(\Lambda^{-1\nu}{}_\lambda p^\lambda), \tag{3.5}$$

where S is the four-component spin representative of s [see (2.10)], $\Lambda^{-1\nu}{}_\lambda$ is related to S by (2.11), and ζ'^i is related to $p'^\nu \equiv \Lambda^{-1\nu}{}_\lambda p^\lambda$ by (3.4). The combination

$$W(s,\mathbf{p}) \equiv e^{-(\zeta^i/2)\alpha_i} S e^{(\zeta'^i/2)\alpha_i} = e^{-\gamma(\boldsymbol{\omega}/2)\cdot\boldsymbol{\alpha}} \tag{3.6}$$

is commonly referred to as a Wigner rotation with $\boldsymbol{\omega}(s,\mathbf{p})$ being the three rotation angles and is represented by a direct sum of two SU_2 rotations. With respect to the z_Ω basis of (2.15) we have the matrix representation,

$$\begin{aligned} W(s,\mathbf{p}) z_\Omega &= z_\Lambda (W)^\Lambda_\Omega, \\ (W) &= \begin{pmatrix} D^{1/2} & 0 \\ 0 & \sigma_1 D^{1/2} \sigma_1 \end{pmatrix}, \\ D^{1/2} &\equiv (e^{-i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}/2}). \end{aligned} \tag{3.7}$$

Here $\boldsymbol{\sigma}$ are the 2×2 Pauli matrices. For $u(\phi) \in U_1$, the action is

$$[U_{(0,u)}\psi](p^\nu) = \psi(p^\nu) e^{\gamma\phi}. \tag{3.8}$$

The action of parity P is identical to (3.5) [see (2.21)],

$$[U_{(0,\mathsf{P})}\psi](p^0,\mathbf{p}) = \pm \gamma_0 \psi(p^0, -\mathbf{p}). \tag{3.9}$$

The action of time reversal T follows (2.21),

$$\begin{aligned} [A_{(0,\mathsf{T})}\psi](p^0,\mathbf{p}) &\equiv \tilde{\psi}^\ddagger(p^0,\mathbf{p}) C^{-1} e^{\phi_T \gamma \alpha_3}, \\ &= -\gamma_0 \psi(p^0, -\mathbf{p}) \gamma_2 e^{\phi_T \gamma \alpha_3}, \end{aligned} \tag{3.10}$$

as does the action of charge conjugation C,

$$\begin{aligned} [A_{(0,\mathsf{C})}\psi](p^\nu) &\equiv \gamma \psi(p^\nu) \gamma_0 C^{-1} e^{\phi_C \gamma \alpha_3} \\ &= -\gamma \psi(p^\nu) \gamma_2 e^{\phi_C \gamma \alpha_3}. \end{aligned} \tag{3.11}$$

Completing the induced representation, we use the character and have for translation $r \in R^4$,

$$\begin{aligned} [U_{(r,I)}\psi](p^\nu) &= [(I+\gamma_0)/2] \psi(p^\nu) e^{-\gamma p_\mu r^\mu} \\ &\quad + [(I-\gamma_0)/2] \psi(p^\nu) e^{+\gamma p_\mu r^\mu}, \end{aligned} \tag{3.12}$$

i.e., the two parity components are phase rotated oppositely by a translation.

The Hilbert space inner product on $\mathscr{H}$ for which (3.5)–(3.9), and (3.12) are unitary and (3.10) and (3.11) are antiunitary is

$$\begin{aligned} &\langle\langle \psi,\phi \rangle\rangle_{\mathscr{H}} \equiv \int dp \langle\langle \psi(p),\phi(p) \rangle\rangle, \\ &\text{where} \\ &dp \equiv (2\pi)^{-3} (mc/p^0) d^3p, \end{aligned} \tag{3.13}$$

and where $\langle\langle \psi(p),\phi(p) \rangle\rangle$ is defined in (2.26). The invariant volume element is dp and the integration domain is the entire upper mass shell. To make clear the details of Wigner's induced unitary representation as well as to proceed with Segal quantization in Sec. IV, we introduce the basis functions $a_{A,\mathbf{q}}$ and $c_{A,\mathbf{q}}$:

$$a_{A,\mathbf{q}}(\mathbf{p}) \equiv (2\pi)^3 (q_0/mc)\delta^3(\mathbf{p}-\mathbf{q}) z_A,$$
$$c_{A,\mathbf{q}}(\mathbf{p}) \equiv (2\pi)^3 (q_0/mc)\delta^3(\mathbf{p}-\mathbf{q}) z_{A+2}, \qquad (3.14)$$

where $q_0 \equiv +[m^2c^2 + \mathbf{q}\cdot\mathbf{q}]^{1/2}$ and $A = \{1,2\}$. The normalization has been chosen so that

$$\langle\langle a_{A,\mathbf{q}}, a_{B,\mathbf{p}}\rangle\rangle_{\mathscr{H}} = (2\pi)^3 (q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p}),$$
$$\langle\langle c_{A,\mathbf{q}}, c_{B,\mathbf{p}}\rangle\rangle_{\mathscr{H}} = (2\pi)^3 (q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p}),$$
$$\langle\langle a_{A,\mathbf{q}}, c_{B,\mathbf{p}}\rangle\rangle_{\mathscr{H}} = 0. \qquad (3.15)$$

The above group actions (3.8)–(3.12) on $a_{A,\mathbf{q}}$ and $c_{A,\mathbf{q}}$ are

$$U_{(0,s)}\, a_{A,\mathbf{q}} = a_{B,\Lambda\mathbf{q}} D^{(1/2)B}{}_A [\omega(s,\Lambda\mathbf{q})],$$
$$U_{(0,s)}\, c_{A,\mathbf{q}} = c_{D,\Lambda\mathbf{q}} \sigma^E_{1C} D^{(1/2)C}{}_B [\omega(s,\Lambda\mathbf{q})] \sigma^B_{1A}, \qquad (3.16)$$

where $j = \frac{1}{2}$ representation matrices $D^{1/2}$ are defined by (3.6) and (3.7):

$$U_{(0,u)}\, a_{A,\mathbf{q}} = a_{A,\mathbf{q}} e^{+\gamma\phi}, \quad A_{(0,\mathrm{T})}\, a_{A,\mathbf{q}} = a_{B,-\mathbf{q}} \sigma^B_{2A} e^{+\gamma\phi_T},$$
$$U_{(0,u)}\, c_{A,\mathbf{q}} = c_{A,\mathbf{q}} e^{+\gamma\phi}, \quad A_{(0,\mathrm{T})}\, c_{A,\mathbf{q}} = -c_{B,-\mathbf{q}} \sigma^B_{2A} e^{+\gamma\phi_T}, \qquad (3.17)$$
$$U_{(0,\mathrm{P})}\, a_{A,\mathbf{q}} = \pm a_{A,-\mathbf{q}}, \quad A_{(0,\mathrm{C})}\, a_{A,\mathbf{q}} = -c_{B,-\mathbf{q}} \sigma^B_{3A} e^{+\gamma\phi_C},$$
$$U_{(0,\mathrm{P})}\, c_{A,\mathbf{q}} = \mp c_{A,-\mathbf{q}}, \quad A_{(0,\mathrm{C})}\, c_{A,\mathbf{q}} = -a_{B,-\mathbf{q}} \sigma^B_{3A} e^{+\gamma\phi_C},$$

follow from (2.22) and complete the unitary and projective antiunitary actions of the homogeneous group on the basis functions we use for $\mathscr{H}$. Now,

$$U_{(r,I)}\, a_{A,\mathbf{q}} = a_{A,\mathbf{q}} e^{-\gamma q_\mu r^\mu}, \quad U_{(r,I)}\, c_{A,\mathbf{q}} = c_{A,\mathbf{q}} e^{+\gamma q_\mu r^\mu}, \qquad (3.18)$$

give the unitary actions of the translations.

The action of the isotropy subgroup $H_x \cong H$ of the point $x \in M^4$ on the basis $a_{A,\mathbf{q}}, c_{A,\mathbf{q}}$, see (2.31), can easily be constructed by applying a homogeneous transformation $h \in H \cong H_{x=0}$, e.g., (3.16) and (3.17) followed by (3.18) with $r = x - h(x)$.

IV. SEGAL QUANTIZING WIGNER'S INDUCED REPRESENTATION

In this section we construct the complex Clifford algebra $\mathscr{C}$ associated with the Hilbert space $\mathscr{H}$ in Sec. III. This is the algebra of annihilation and creation operators of Dirac's electron–positron theory. To construct this algebra we follow the procedure described by Shale and Stinespring and frequently called Segal quantization.[15–21,28–30] The procedure starts by identifying the complex space $\mathscr{H}$ with a real Hilbert space $\mathscr{H}_R$ possessing a symmetric inner product, followed by the construction of its associated Clifford algebra $\mathscr{C}_R$ according to the prescription given in (2.1). This real infinite-dimensional Clifford algebra, when complexified, becomes the desired operator algebra $\mathscr{C}$. Because this is the second vector space→Clifford algebra construction required to obtain a quantum field theory of electrons and positrons, we call it second Cliffordization.

In the first step of second Cliffordization, complex vectors ψ, $\phi \in \mathscr{H}$, are mapped, respectively, one-to-one onto real vectors $\psi_R, \phi_R \in \mathscr{H}_R$ in such a manner as to relate real and complex inner products by

$$(\phi_R, \psi_R)_R \equiv (\langle\langle\phi,\psi\rangle\rangle_{\mathscr{H}} + \langle\langle\psi,\phi\rangle\rangle_{\mathscr{H}})/2. \qquad (4.1)$$

Multiplying $\psi \in \mathscr{H}$ by the unit imaginary number γ does not give an independent vector $\psi\gamma \in \mathscr{H}$; however, their images in $\mathscr{H}_R, \psi_R$, and $(\psi\gamma)_R$ are not only independent but, according to (4.1) are orthogonal. Multiplication by γ in $\mathscr{H}$ induces a complex structure on $\mathscr{H}_R$, $\gamma_R | \mathscr{H}_R \to \mathscr{H}_R$ defined by

$$\gamma_R[\psi_R] \equiv (\psi\gamma)_R, \qquad (4.2)$$

and satisfies the required $\gamma_R\gamma_R = -I$. Every complex linear transformation of $\mathscr{H}$ induces a corresponding real linear transformation $\mathscr{H}_R$ that commutes with this complex structure. In particular the unitary invariance group $\mathbf{U}_{\mathscr{H}}$ of $\langle\langle,\rangle\rangle_{\mathscr{H}}$ is easily seen to be isomorphic to the subgroup of $(,)_R$ invariant orthogonal transformations $\mathbf{Q}_{R,\gamma}$ that commute with γ_R, i.e., every $U_{\mathscr{H}} \in \mathbf{U}_{\mathscr{H}}$ satisfies $U_{\mathscr{H}} \to O_R$ for some O_R provided

$$(Q_R\phi_R, O_R\psi_R)_R = (\psi_R, \phi_R)_{,R} \quad \text{for all } \phi_R, \psi_R \in \mathscr{H}_R$$

and

$$\gamma_R O_R = O_R \gamma_R, \quad \text{i.e., provided } O_R \in \mathbf{O}_{R,\gamma}. \qquad (4.3)$$

Notice that $\gamma \leftrightarrow \gamma_R$ belongs to this isomorphism. In terms of the basis functions (3.14) of $\mathscr{H}$, the one-to-one $\mathscr{H} \leftrightarrow \mathscr{H}_R$ mapping appears as

$$a_{A,\mathbf{q}} \leftrightarrow a_{{}_RA,\mathbf{q}}, \quad a_{A,\mathbf{q}}\gamma \leftrightarrow \gamma_R[a_{{}_RA,\mathbf{q}}],$$
$$c_{A,\mathbf{q}} \leftrightarrow c_{{}_RA,\mathbf{q}}, \quad c_{A,\mathbf{q}}\gamma \leftrightarrow \gamma_R[c_{{}_RA,\mathbf{q}}], \qquad (4.4)$$

and from (3.15) and (4.1) the symmetric real inner product of basis vectors becomes

$$(a_{{}_RA,\mathbf{q}}, a_{{}_RB,\mathbf{p}})_R = (c_{{}_RA,\mathbf{q}}, c_{{}_RB,\mathbf{p}})_R = (2\pi)^3 (q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p}),$$
$$(\gamma_R[a_{{}_RA,\mathbf{q}}], \gamma_R[a_{{}_RB,\mathbf{p}}])_R = (\gamma_R[c_{{}_RA,\mathbf{q}}], \gamma_R[c_{{}_RB,\mathbf{p}}])_R = (2\pi)^3 (q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p}),$$
$$(a_{{}_RA,\mathbf{q}}, c_{{}_RB,\mathbf{p}})_R = (\gamma_R[a_{{}_RA,\mathbf{q}}], \gamma_R[c_{{}_RB,\mathbf{p}}])_R = \text{etc.} = 0. \qquad (4.5)$$

The real vector space $\mathscr{H}_R$ plays the same role as the four-dimensional Lorentz inner product tangent space of M^4 plays in Sec. II, and the above orthonormal basis plays the role of the e_μ. The unitary representation $P \to \mathbf{U}_{\mathscr{H}}$, of the inhomogeneous group (2.29), whose action on the basis vectors (3.14) of $\mathscr{H}$ is given by (3.16) to (3.18) and preserves (3.15), would now appear as an orthogonal representation appropriately transforming the basis vectors (4.4) while preserving (4.5).

Following (2.1) an infinite-dimensional Clifford algebra $\mathscr{C}_R$ can be constructed from $\mathscr{H}_R$. We write the isomorphic image of $\mathscr{H}_R$ in $\mathscr{C}_R$ as $\mathscr{C}^1_R$, and write the image vectors in boldface rather than with a "1" superscript as in (2.1). For example, the $\mathscr{H}_R \leftrightarrow \mathscr{C}^1_R$ mapping of basis vectors is written as

$$a_{{}_RA,\mathbf{q}} \leftrightarrow \mathbf{a}_{A,\mathbf{q}}, \quad \gamma_R[a_{{}_RA,\mathbf{q}}] \leftrightarrow \gamma[\mathbf{a}_{A,\mathbf{q}}],$$
$$c_{{}_RA,\mathbf{q}} \leftrightarrow \mathbf{c}_{A,\mathbf{q}}, \quad \gamma_R[c_{{}_RA,\mathbf{q}}] \leftrightarrow \gamma[\mathbf{c}_{A,\mathbf{q}}]. \qquad (4.6)$$

This basis identification is equivalent to $e_\mu \leftrightarrow \gamma_\mu$ for M^4. The Clifford algebra $\mathscr{C}_R$ is generated by all real linear combinations of products of $\mathscr{C}^1_R$ basis vectors, $\{\mathbf{a}_{A,\mathbf{q}}, \mathbf{c}_{A,\mathbf{q}}, \gamma[\mathbf{a}_{A,\mathbf{q}}], \gamma[\mathbf{c}_{A,\mathbf{q}}]\}$, and can be expressed as

$$\mathscr{C}_R = \mathscr{C}^0_R \oplus \mathscr{C}^1_R \oplus \mathscr{C}^2_R \oplus \mathscr{C}^3_R \oplus \cdots, \qquad (4.7)$$

where $\mathscr{C}^0_R$ stands for all real multiples of the identity $\mathscr{I}$, $\mathscr{C}^1_R$

for all real linear combinations of basis vectors $\{\mathbf{a}_{A,\mathbf{q}},\mathbf{c}_{A,\mathbf{q}},\gamma[\mathbf{a}_{A,\mathbf{q}}],\gamma[\mathbf{c}_{A,\mathbf{q}}]\}$, $\mathscr{C}^2_R$ for all real linear combinations of products of pairs of basis vectors with no two pairs being equal, e.g., $\mathbf{a}_{A,\mathbf{q}}\mathbf{a}_{B,\mathbf{p}}$ where $A\neq B$ or $\mathbf{q}\neq\mathbf{p}$, $\mathscr{C}^3_R$ for all real linear combinations of products of triples of basis vectors with no two being equal $\cdots$, etc. Linear combinations include integrations over the continuous mass shell indices $\mathbf{q}$, $\mathbf{p}$, etc. The Clifford algebraic multiplication constraint (iii) of (2.1), evaluated using (4.5), appears as

$$\begin{aligned}\mathbf{a}_{A,\mathbf{q}}\mathbf{a}_{B,\mathbf{p}}+\mathbf{a}_{B,\mathbf{p}}\mathbf{a}_{A,\mathbf{q}}&=\gamma[\mathbf{a}_{A,\mathbf{q}}]\gamma[\mathbf{a}_{B,\mathbf{p}}]+\gamma[\mathbf{a}_{B,\mathbf{p}}]\gamma[\mathbf{a}_{A,\mathbf{q}}]\\&=2(2\pi)^3(q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p})\mathscr{I},\\\mathbf{a}_{A,\mathbf{q}}\mathbf{c}_{B,\mathbf{p}}+\mathbf{c}_{B,\mathbf{p}}\mathbf{a}_{A,\mathbf{q}}&=\gamma[\mathbf{a}_{A,\mathbf{q}}]\gamma[\mathbf{c}_{B,\mathbf{p}}]+\gamma[\mathbf{c}_{B,\mathbf{p}}]\gamma[\mathbf{a}_{A,\mathbf{q}}]\\&=\text{etc.}=0.\end{aligned}\tag{4.8}$$

Because $\mathscr{H}_R$ is isomorphic to $\mathscr{C}^1_R$ every orthogonal transformation of $\mathscr{H}_R$ in $\mathbf{O}_{R,\gamma}$ corresponds to an orthogonal transformation of $\mathscr{C}^1_R$. In particular, the complex structure mapping $\gamma_R|\mathscr{H}_R\to\mathscr{H}_R$ corresponds to the mapping $\gamma|\mathscr{C}^1_R\to\mathscr{C}^1_R$ as indicated in (4.6). Since $\mathscr{C}^1_R$ generates $\mathscr{C}_R$, an orthogonal transformation O_R of $\mathscr{C}^1_R$ can be extended to an algebra automorphism $O\in\mathbf{O}_\gamma\subset\mathrm{Aut}(\mathscr{C}_R)$ of $\mathscr{C}_R$ by simply requiring $O(\mathbf{c}_1\mathbf{c}_2)=O(\mathbf{c}_1)O(\mathbf{c}_2)$ and $O(\mathbf{c}^1)=O_R(\mathbf{c}^1)$ when $\mathbf{c}^1\in\mathscr{C}^1_R$. As an example, the γ mapping extends to all of $\mathscr{C}_R$ as an algebra automorphism; e.g., when γ is applied to the identity $\mathscr{I}$ of $\mathscr{C}_R$, it gives $\mathscr{I}$ back. Consequently, γ does not satisfy the required $\gamma\gamma=-I$ to provide a complex structure for all of $\mathscr{C}_R$. However, $\mathscr{C}_R$ can be complexified by taking complex (rather than real) linear combinations as in (4.7); the resulting complex algebra $\mathscr{C}$ turns out to be the desired electron–positron operator algebra,

$$\mathscr{C}=\mathscr{C}^0\oplus\mathscr{C}^1\oplus\mathscr{C}^2\oplus\mathscr{C}^3\oplus\cdots.\tag{4.9}$$

We will simply denote this complex structure by multiplying by the unit imaginary number "i" and * as complex conjugation. The automorphisms $\mathbf{O}_\gamma$ of $\mathscr{C}_R$ can be extended to $\mathscr{C}$ by simply requiring that they commute with the new complex structure. In particular, γ extends from $\mathscr{C}_R$ to $\mathscr{C}$ by requiring $\gamma i=i\gamma$. Here $\mathscr{C}^1$ represents the space of all one-particle (electron–positron) annihilation and creation operators. Rather than using $\{\mathbf{a}_{B,\mathbf{p}},\mathbf{c}_{B,\mathbf{p}},\gamma[\mathbf{a}_{B,\mathbf{p}}],\gamma[\mathbf{c}_{B,\mathbf{p}}]\}$ as basis vectors, the more conventional set $\{\mathbf{b}_{B\mathbf{p}},\mathbf{b}^\dagger_{B\mathbf{p}},\mathbf{d}_{B\mathbf{p}},\mathbf{d}^\dagger_{B\mathbf{p}}\}$ can be used,

$$\begin{aligned}&\mathbf{b}_{B\mathbf{p}}\equiv\tfrac{1}{2}(\mathbf{a}_{B,\mathbf{p}}-i\gamma[\mathbf{a}_{B,\mathbf{p}}]),\quad \mathbf{b}^\dagger_{B\mathbf{p}}\equiv\tfrac{1}{2}(\mathbf{a}_{B,\mathbf{p}}+i\gamma[\mathbf{a}_{B,\mathbf{p}}]),\\&\mathbf{d}^\dagger_{B\mathbf{p}}\equiv\tfrac{1}{2}(\mathbf{c}_{B,\mathbf{p}}-i\gamma[\mathbf{c}_{B,\mathbf{p}}]),\quad \mathbf{d}_{B\mathbf{p}}\equiv\tfrac{1}{2}(\mathbf{c}_{B,\mathbf{p}}+i\gamma[\mathbf{c}_{B,\mathbf{p}}]).\end{aligned}\tag{4.10}$$

The conjugate linear mapping $^\dagger|\mathscr{C}^1\to\mathscr{C}^1$ defined by (4.10) acts as the identity on $\mathscr{C}^1_R$ commuting with γ but anti commuting with "i" multiplication. Because $\mathscr{C}^1$ generates $\mathscr{C}$, the † mapping can be immediately extended to a unique antiautomorphism of $\mathscr{C}$ by requiring that

$$\begin{aligned}&(\mathbf{c}_1+\mathbf{c}_2)^\dagger=\mathbf{c}_1^\dagger+\mathbf{c}_2^\dagger,\\&(\mathbf{c}_1\mathbf{c}_2)^\dagger=\mathbf{c}_2^\dagger\mathbf{c}_1^\dagger,\quad \mathbf{c}_1\mathbf{c}_2\in\mathscr{C},\\&\mathbf{c}_0^\dagger=(r+\overset{*}{i}r')\mathscr{I},\quad \mathbf{c}_0=(r+ir')\mathscr{I}\in\mathscr{C}^0.\end{aligned}\tag{4.11}$$

The fundamental antiautomorphism † is just the analog of reversion (2.5) for real Clifford algebras. The subset of self-conjugate elements ($\mathbf{c}^\dagger=\mathbf{c}$) is just those identified with the original Hilbert space $\mathscr{H}\cong\mathscr{H}_R$. A complex algebra such as $\mathscr{C}$ with such an involuting, $(\mathbf{c}^\dagger)^\dagger=\mathbf{c}$, antiautomorphism is called a $\star$ algebra and with appropriate norm and completion becomes a $C\star$ algebra.[31] To make contact with the "CAR" algebra construction approach, one has only to complexify $\mathscr{H}_R$ with imaginary unit "i" by identifying it with $\mathscr{C}^1$. Then $\mathscr{C}$ is the CAR algebra of this complex Hilbert space.

In (4.10) careful attention must be paid to the indices $A,B=\{1,2\}$ being up or down. This index must be raised and lowered with the Pauli spinor metric, e.g., with

$$\begin{aligned}&\epsilon^{AB}=\epsilon_{AB}\equiv i\sigma^A_{2B}=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\\&\epsilon^{AC}\epsilon_{BC}=\delta^A_B.\end{aligned}\tag{4.12}$$

The operators appearing in the Dirac fields $\Psi(x)$ and $\Psi^\dagger(x)$ are with the "A" index "up,"

$$\begin{aligned}&\mathbf{b}^A_\mathbf{p}\equiv\epsilon^{AB}\mathbf{b}_{B\mathbf{p}},\quad \mathbf{b}^{A\dagger}_\mathbf{p}\equiv\epsilon^{AB}\mathbf{b}^\dagger_{B\mathbf{p}},\\&\mathbf{b}_{B\mathbf{p}}=\epsilon_{AB}\mathbf{b}^A_\mathbf{p},\quad \mathbf{b}^\dagger_{B\mathbf{p}}=\epsilon_{AB}\mathbf{b}^{A\dagger}_\mathbf{p},\quad\text{etc.},\end{aligned}\tag{4.13}$$

and from (4.8) satisfy the required anticommutation relations,

$$\begin{aligned}\mathbf{b}^A_\mathbf{q}\mathbf{b}^{B\dagger}_\mathbf{p}+\mathbf{b}^{B\dagger}_\mathbf{p}\mathbf{b}^A_\mathbf{q}&=\mathbf{d}^A_\mathbf{q}\mathbf{d}^{B\dagger}_\mathbf{p}+\mathbf{d}^{B\dagger}_\mathbf{p}\mathbf{d}^A_\mathbf{q}\\&=(2\pi)^3(q_0/mc)\delta_{AB}\delta^3(\mathbf{q}-\mathbf{p})\mathscr{I},\\\mathbf{b}^A_\mathbf{q}\mathbf{b}^B_\mathbf{p}+\mathbf{b}^B_\mathbf{p}\mathbf{b}^A_\mathbf{q}&=\mathbf{d}^A_\mathbf{q}\mathbf{d}^B_\mathbf{p}+\mathbf{d}^B_\mathbf{p}\mathbf{d}^A_\mathbf{q}=\text{etc.}=0.\end{aligned}\tag{4.14}$$

The γ action on the conventional basis (4.10) is (using $\gamma\gamma=-I$ on $\mathscr{C}^1$)

$$\begin{aligned}&\gamma[\mathbf{b}_{B\mathbf{p}}]=i\mathbf{b}_{B\mathbf{p}},\quad \gamma[\mathbf{d}^\dagger_{B\mathbf{p}}]=i\mathbf{d}^\dagger_{B\mathbf{p}},\\&\gamma[\mathbf{b}^\dagger_{B\mathbf{p}}]=-i\mathbf{b}^\dagger_{B\mathbf{p}},\quad \gamma[\mathbf{d}_{B\mathbf{p}}]=-i\mathbf{d}_{B\mathbf{p}}.\end{aligned}\tag{4.15}$$

We are now in a position to compute explicitly the extension of the inhomogeneous group's unitary action $\mathbf{U}_{\mathscr{P}}$ (3.16)–(3.18) on $\mathscr{H}$ to an $\mathbf{O}_\gamma$ action on $\mathscr{C}^1$. By applying $U_{(0,s)}$ of (3.16) to (4.10) and using (4.13), the identity component of the homogeneous group acts according to

$$\begin{aligned}O_{(0,s)}\mathbf{b}_{A\mathbf{q}}&=D^{(1/2)}(\boldsymbol{\omega})^B{}_A\mathbf{b}_{B\Lambda\mathbf{q}},\Leftrightarrow O_{(0,s)}\mathbf{b}^A_\mathbf{q}\\&=D^{(1/2)}(-\boldsymbol{\omega})^A{}_B\mathbf{b}^B_{\Lambda\mathbf{q}},\\O_{(0,s)}\mathbf{d}^\dagger_{A\mathbf{q}}&=\sigma^D_{1C}D^{(1/2)}(\boldsymbol{\omega})^C{}_B\sigma^B_{1A}\mathbf{d}^\dagger_{D\Lambda\mathbf{q}},\Leftrightarrow O_{(0,s)}\mathbf{d}^{A\dagger}_\mathbf{q}\\&=\sigma^A_{1B}D^{(1/2)}(-\boldsymbol{\omega})^B{}_C\sigma^C_{1D}\mathbf{d}^{D\dagger}_{\Lambda\mathbf{q}},\end{aligned}$$

etc., where

$$D^{(1/2)}(\boldsymbol{\omega})^B_A\equiv D^{(1/2)B}{}_A[\boldsymbol{\omega}(s,\Lambda\mathbf{q})]$$

satisfies

$$(\sigma_2D^{1/2}(\boldsymbol{\omega})\sigma_2)^B{}_A=\overset{*}{D}{}^{1/2}(\boldsymbol{\omega})^B{}_A=D^{1/2}(-\boldsymbol{\omega})^A{}_B.\tag{4.16}$$

We have used O's to represent these linear transformations because they are orthogonal on $\mathscr{C}^1_R$ (and would be real unitary on $\mathscr{C}^1$ if given the obvious Hilbert space inner product). The $j=\frac{1}{2}$ unitary representation matrices $D^{1/2}$ are defined by (3.6) and (3.7). The U_1 homogeneous transformations from (3.17) act simply as

$$O_{(0,u)}\,\mathbf{b}_\mathbf{q}^A = e^{+i\phi}\mathbf{b}_\mathbf{q}^A, \quad O_{(0,u)}\,\mathbf{b}_\mathbf{q}^{A\dagger} = e^{-i\phi}\mathbf{b}_\mathbf{q}^{A\dagger},$$
$$O_{(0,u)}\,\mathbf{d}_\mathbf{q}^{A\dagger} = e^{+i\phi}\mathbf{d}_\mathbf{q}^{A\dagger}, \quad O_{(0,u)}\,\mathbf{d}_\mathbf{q}^A = e^{-i\phi}\mathbf{d}_\mathbf{q}^A. \tag{4.17}$$

Parity, time reversal, and charge conjugation from (3.17) extend to

$$O_{(0,\mathrm{P})}\,\mathbf{b}_\mathbf{q}^A = \pm\,\mathbf{b}_{-\mathbf{q}}^A, \quad O_{(0,\mathrm{P})}\,\mathbf{b}_\mathbf{q}^{A\dagger} = \pm\,\mathbf{b}_{-\mathbf{q}}^{A\dagger},$$
$$O_{(0,\mathrm{P})}\,\mathbf{d}_\mathbf{q}^A = \mp\,\mathbf{d}_{-\mathbf{q}}^A, \quad O_{(0,\mathrm{P})}\,\mathbf{d}_\mathbf{q}^{A\dagger} = \mp\,\mathbf{d}_{-\mathbf{q}}^{A\dagger},$$
$$A_{(0,\mathrm{T})}\,\mathbf{b}_\mathbf{q}^A = \sigma_{2B}^A e^{+i\phi_T}\mathbf{b}_{-\mathbf{q}}^B, \quad A_{(0,\mathrm{T})}\,\mathbf{b}_\mathbf{q}^{A\dagger} = \overset{*}{\sigma}{}_{2B}^A e^{-i\phi_T}\mathbf{b}_{-\mathbf{q}}^{B\dagger}, \tag{4.18}$$
$$A_{(0,\mathrm{T})}\,\mathbf{d}_\mathbf{q}^A = \sigma_{2B}^A e^{-i\phi_T}\mathbf{d}_{-\mathbf{q}}^B, \quad A_{(0,\mathrm{T})}\,\mathbf{d}_\mathbf{q}^{A\dagger} = \overset{*}{\sigma}{}_{2B}^A e^{+i\phi_T}\mathbf{d}_{-\mathbf{q}}^{B\dagger},$$
$$O_{(0,\mathrm{C})}\,\mathbf{b}_\mathbf{q}^A = \sigma_{3B}^A e^{-i\phi_C}\mathbf{d}_\mathbf{q}^B, \quad O_{(0,\mathrm{C})}\,\mathbf{b}_\mathbf{q}^{A\dagger} = \sigma_{3B}^A e^{+i\phi_C}\mathbf{d}_\mathbf{q}^{B\dagger},$$
$$O_{(0,\mathrm{C})}\,\mathbf{d}_\mathbf{q}^A = \sigma_{3B}^A e^{+i\phi_C}\mathbf{b}_\mathbf{q}^B, \quad O_{(0,\mathrm{C})}\,\mathbf{d}_\mathbf{q}^{A\dagger} = \sigma_{3B}^A e^{-i\phi_C}\mathbf{b}_\mathbf{q}^{B\dagger},$$

completing the orthgonal and projective antiorthogonal actions of the homogeneous group on the basis for $\mathscr{C}^1$. Notice that P and C actions commute with the new complex structure, "i" multiplication, but the T action has to be taken to anticommute with it. For this reason the action of C in (3.17) has been changed from $A_{(0,\mathrm{C})} \to O_{(0,\mathrm{C})}$ while the T action remains conjugate–linear and written as $A_{(0,\mathrm{T})}$ in (4.18). From (3.18) the translations act on $\mathscr{C}^1$ by

$$O_{(r,I)}\,\mathbf{b}_\mathbf{q}^A = e^{-iq_\mu r^\mu}\mathbf{b}_\mathbf{q}^A, \quad O_{(r,I)}\,\mathbf{b}_\mathbf{q}^{A\dagger} = e^{+iq_\mu r^\mu}\mathbf{b}_\mathbf{q}^{A\dagger},$$
$$O_{(r,I)}\,\mathbf{d}_\mathbf{q}^A = e^{-iq_\mu r^\mu}\mathbf{d}_\mathbf{q}^A, \quad O_{(r,I)}\,\mathbf{d}_\mathbf{q}^{A\dagger} = e^{+iq_\mu r^\mu}\mathbf{d}_\mathbf{q}^{A\dagger}. \tag{4.19}$$

The above linear orthogonal transformations $\mathbf{O}_\gamma$ are analogous to the linear Lorentz transformations (Λ_μ^ν) of (2.11) for the four-dimensional M^4 space. We now look for the equivalent of the $\mathrm{Pin}_{1,3}$ group, i.e., the group $\mathscr{U}$ defined by

$$U\in\mathscr{U} \Leftrightarrow U\in\mathscr{C},$$
$$U^\dagger U = \mathscr{I},$$

and

$$U\mathscr{C}^1U^\dagger = \mathscr{C}^1. \tag{4.20}$$

The Pin covering of the orthogonal group $\mathscr{U}\to\mathbf{O}_\gamma$ is defined analogous to (2.11) by

$$U\mathbf{c}^1U^\dagger = O\mathbf{c}^1, \quad \mathbf{c}^1\in\mathscr{C}^1, \tag{4.21}$$

and has a kernel $\cong U_1$, i.e., $e^{i\phi}\mathscr{I}\to I$. The complex Clifford algebra $\mathscr{C}$ can be thought of as a Hilbert space by defining a Hermitian inner product

$$\langle \mathbf{c}_1,\mathbf{c}_2\rangle = (\mathbf{c}_1^\dagger\mathbf{c}_2)_\mathscr{I}, \tag{4.22}$$

where $(\ )_\mathscr{I}$ means the component of ( ) contained in $\mathscr{C}^0\propto\mathscr{I}$. In this way $\mathscr{C}$ can be thought of as a direct sum of Hilbert spaces (4.9), of which $\mathscr{C}^0$ is of dimension 1. The group $\mathscr{U}$ acting as a group of inner automorphisms (vector action) as in (4.20), acts unitarily on each $\mathscr{C}^k$ separately. However, as a spinor action on the left, $\mathbf{c}\to U\mathbf{c}$, $\mathscr{U}$ acts unitarily mixing the $\mathscr{C}^k$. The spin representation of this group is found precisely as we found the four-component Dirac spinors, by finding a primitive idempotent, i.e., a projection operator.[32,33] One choice of the idempotent is

$$\mathscr{P} \equiv \lim_{i\to\mathbf{p}} \prod_{A,i} \mathbf{b}_i^A\mathbf{b}_i^{A\dagger}\mathbf{d}_i^A\mathbf{d}_i^{A\dagger}, \tag{4.23}$$

with the properties that

$$\mathscr{P}^2 = \mathscr{P}, \quad \mathscr{P}^\dagger = \mathscr{P}. \tag{4.24}$$

Note that we have used a discrete label "i" in place of $\mathbf{p}$ and obtain the continuum by a limiting procedure. We assume that this process can be made rigorous;[34] however, for our purpose the reader can take

$$\mathbf{b}_i^A \equiv \frac{1}{\sqrt{V}}\int_{D_i} dp\,\mathbf{b}_\mathbf{p}^A, \tag{4.25}$$

where the domains D_i are disjoint, completely cover the upper mass shell, and have invariant volume V. The $\mathbf{b}_i^{A\dagger}$, $\mathbf{d}_i^A$, and $\mathbf{d}_i^{A\dagger}$ are similarly defined and satisfy the expected anticommutation relations, e.g.,

$$\mathbf{b}_i^{A\dagger}\mathbf{b}_j^B + \mathbf{b}_j^B\mathbf{b}_i^{A\dagger} = \delta_{ij}^{AB}\mathscr{I}. \tag{4.26}$$

The important property of $\mathscr{P}$ is that

$$\mathbf{b}_\mathbf{p}^A\mathscr{P} = \mathbf{d}_\mathbf{p}^A\mathscr{P} = 0. \tag{4.27}$$

The Fock vacuum state $|0\rangle$, is defined almost identically to $\mathscr{P}$ except appropriately normalized,

$$|0\rangle \equiv \lim_{i\to\mathbf{p}} \prod_{A,i} 2\mathbf{b}_i^A\mathbf{b}_i^{A\dagger}\mathbf{d}_i^A\mathbf{d}_i^{A\dagger}. \tag{4.28}$$

The corresponding minimal left ideal $\mathscr{C}\mathscr{P}$ would be called a spinor space in analogy with (2.12) but is commonly called Fock space. It is generated by multiplying $\mathscr{P}$ or $|0\rangle$ on the left by $\mathscr{C}$ and is spanned by the following basis states:

$$\{|0\rangle, \mathbf{b}_\mathbf{p}^{A\dagger}|0\rangle, \mathbf{d}_\mathbf{p}^{A\dagger}|0\rangle, \mathbf{b}_\mathbf{p}^{A\dagger}\mathbf{b}_\mathbf{q}^{B\dagger}|0\rangle,$$
$$\mathbf{d}_\mathbf{p}^{A\dagger}\mathbf{d}_\mathbf{q}^{B\dagger}|0\rangle, \mathbf{b}_\mathbf{p}^{A\dagger}\mathbf{d}_\mathbf{q}^{B\dagger}|0\rangle, \cdots\}, \tag{4.29}$$

where either $A\neq B$ or $\mathbf{p}\neq\mathbf{q}$, etc. The normalization of the vacuum state is checked using (4.22),

$$\langle 0|0\rangle \equiv (|0\rangle^\dagger|0\rangle)_\mathscr{I} = \left(\lim_{i\to\mathbf{p}} \prod_{A,i} 4\mathbf{b}_i^A\mathbf{b}_i^{A\dagger}\mathbf{d}_i^A\mathbf{d}_i^{A\dagger}\right)_\mathscr{I} = 1. \tag{4.30}$$

The steps in (4.30) require using the idempotentcy of $\mathbf{b}_i^A\mathbf{b}_i^{A\dagger}$ and $\mathbf{d}_i^A\mathbf{d}_i^{A\dagger}$ and decomposing $\mathbf{b}_i^A = (\mathbf{a}_i^A - i\gamma[\mathbf{a}_i^A])/2$ into self-conjugate parts as in (4.10).

So far we have seen how familiar quantities, like the vacuum and the Fock space of a spin-half quantum field theory, emerge naturally from various algebraic quantities, such as an idempotent and its minimal left ideal in second Cliffordization. Since the minimal left ideal $\mathscr{C}\mathscr{P}$ provides a spinor representation of the linear orthogonal transformations as defined in (4.16)–(4.19), we can also see how physical operators on the Fock space emerge as representations of the generators of inhomogeneous transformations. For example, for an infinitesimal translation by an amount ϵ^μ, we have from (4.19)

$$O_{(\epsilon,I)}\,\mathbf{b}_\mathbf{q}^A = e^{-iq_\mu\epsilon^\mu}\mathbf{b}_\mathbf{q}^A \approx (I - iq_\mu\epsilon^\mu)\mathbf{b}_\mathbf{q}^A, \tag{4.31}$$

and from (4.21) the corresponding translation operator is

$$U \equiv e^{i\epsilon_\mu\mathbf{P}^\mu} \approx (\mathscr{I} + i\epsilon_\mu\mathbf{P}^\mu). \tag{4.32}$$

The momentum operators $\mathbf{P}^\mu$ are the representatives of translation generators and are only determined by (4.31) up to a generator of the kernel of the Pin covering as discussed in (4.21),

$$U\mathbf{b}_\mathbf{q}^A U^\dagger = O_{(\epsilon,I)}\mathbf{b}_\mathbf{q}^A$$
$$\Rightarrow [\mathbf{b}_\mathbf{q}^A,\mathbf{P}^\mu] = q^\mu \mathbf{b}_\mathbf{q}^A$$
$$\Rightarrow \mathbf{P}^\mu = \sum_A \int dq\, q^\mu \mathbf{b}_\mathbf{q}^{A\dagger}\mathbf{b}_\mathbf{q}^A + \text{commuting terms.} \quad (4.33)$$

Similar consideration can be carried out for the d's. Then the total momentum operator becomes

$$\mathbf{P}^\mu = \sum_A \int dq\, q^\mu(\mathbf{b}_\mathbf{q}^{A\dagger}\mathbf{b}_\mathbf{q}^A + \mathbf{d}_\mathbf{q}^{A\dagger}\mathbf{d}_\mathbf{q}^A) + (\text{const})\mathscr{I}. \quad (4.34)$$

The arbitrary constant term comes from the generator of the U_1 kernel and can be eliminated by requiring that the vacuum be translationally invariant or equivalently

$$\mathbf{P}^\mu|0\rangle = 0 \quad \Rightarrow \text{const} = 0. \quad (4.35)$$

That is, we require the vacuum state to have zero energy and momentum. This is equivalent to the normal ordering procedure in quantum field theory. The resulting Hamiltonian for free electrons and positrons is

$$\mathbf{H} = c\mathbf{P}^0 = c\sum_A \int dq\, q^0(\mathbf{b}_\mathbf{q}^{A\dagger}\mathbf{b}_\mathbf{q}^A + \mathbf{d}_\mathbf{q}^{A\dagger}\mathbf{d}_q^A), \quad (4.36)$$

where

$$\mathbf{N}_\mathbf{q}^{A\,-} \equiv \mathbf{b}_\mathbf{q}^{A\dagger}\mathbf{b}_\mathbf{q}^A, \quad \mathbf{N}_\mathbf{q}^{A\,+} \equiv \mathbf{d}_\mathbf{q}^{A\dagger}\mathbf{d}_\mathbf{q}^A \quad (4.37)$$

are the number operators for the electrons and the positrons, respectively.

The final physical observable we obtain is the charge operator Q. It comes from the infinitesimal phase transformations (4.17),

$$O_{(0,u)}\mathbf{b}_\mathbf{q}^A = e^{+i\phi}\mathbf{b}_\mathbf{q}^A \approx (I + i\phi)\mathbf{b}_\mathbf{q}^A,$$
$$O_{(0,u)}\mathbf{d}_\mathbf{q}^{A\dagger} = e^{+i\phi}\mathbf{d}_\mathbf{q}^{A\dagger} \approx (I + i\phi)\mathbf{d}_\mathbf{q}^{A\dagger}. \quad (4.38)$$

If the corresponding U of (4.21) is written as

$$U = e^{i\phi\mathbf{Q}} \approx \mathscr{I} + i\phi\mathbf{Q}, \quad (4.39)$$

the reader finds by steps similar to (4.33)–(4.35) but even simpler,

$$\mathbf{Q} = \sum_A \int dq(-\mathbf{b}_\mathbf{q}^{A\dagger}\mathbf{b}_\mathbf{q}^A + \mathbf{d}_\mathbf{q}^{A\dagger}\mathbf{d}_\mathbf{q}^A) + (\text{const})\mathscr{I}. \quad (4.40)$$

Again, the constant term comes from the U_1 kernel and can be eliminated by requiring that the vacuum have no charge,

$$\mathbf{Q}|0\rangle = 0 \Rightarrow \text{const} = 0. \quad (4.41)$$

V. CONCLUSIONS

In this paper we have tried to "tie together" some of the "loose ends" in the free electron–positron field theory by showing how the appropriate construction of two successive Clifford algebras can result in the free quantum field theory. The first Clifford algebra was associated with the tangent space of any point in Minkowski space and its Lorentz invariant inner product. The second was associated with an infinite-dimensional Hilbert space and its Poincaré [enlarged (2.29)] invariant Hermitian inner product, which we constructed (via Wigner's procedure) using the spinor representation of the first Clifford algebra. All elements of the noninteracting theory seem to be accounted for by this "second Cliffordization." In particular, the operator algebra of the free field theory is just the second complex Clifford algebra. The familiar abstract Fock representation appears concretely as a spinor representation space in the infinite dimensional algebra analogous to four-component Dirac spinors in the finite Minkowski algebra. Two obvious extensions of this work are to higher dimensions and to the inclusion of interactions with external fields. Extensions to other spins as well as to massless fields seem straightforward.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy.

The authors wish to thank R. Ablamowicz and S.T. Ali for suggesting improvements in the first version of this manuscript.

[1]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[2]C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[3]W. K. Clifford, Am. J. Math. **1**, 350 (1878).

[4]C. C. Chevalley, *The Algebraic Theory of Spinors* (Columbia Univ., New York, 1954).

[5]M. Riesz, *Clifford Numbers and Spinors*, Lecture Series No. 38 (Institute for Fluid Dynamics and Applied Mathematics, Univ. of Maryland, 1958).

[6]D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966).

[7]P. Budinich and A. Trautman, *The Spinoral Chessboard* (Springer, Berlin, 1988); P. Lounesto, Found. Phys. **11**, 721 (1981).

[8]I. M. Benn and R. W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Hilger, Bristol, 1987).

[9]E. P. Wigner, Ann. Math. **40**, 149 (1939).

[10]E. P. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gursey (Gordon and Breach, New York, 1964).

[11]G. W. Mackey, Ann. Math. **55**, 101 (1952).

[12]F. R. Halpern, *Special Relativity and Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1968).

[13]D. J. Simms, Lectures Notes in Mathematics Vol. 52 (Springer, Berlin, 1968).

[14]Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidal, Dordrecht, 1986).

[15]I. E. Segal, Ann. Math. **48**, 930 (1947).

[16]I. E. Segal, *Mathematical Problems in Relativistic Physics* (Am. Math. Soc., Providence, 1963).

[17]D. Shale and W. F. Stinespring, Ann. Math. **80**, 365 (1964).

[18]D. Shale and W. F. Stinespring, J. Math. Mech. **14**, 315 (1965).

[19]R. F. Streater, Rep. Prog. Phys. **38**, 847 (1975).

[20]J. M. Cook, Trans. Am. Math. Soc. **74**, 222 (1953).

[21]M. Weinless, J. Funct. Anal. **4**, 350 (1969).

[22]V. Fock, Z. Phys. **75**, 622 (1932).

[23]P. Lounesto and E. Latvamaa, Proc. AMS **79**, 533 (1980).

[24]P. Lounesto and G. P. Wene, Acta Appl. Math. **9**, 165 (1987).

[25]A. Crumeyrolle, *Algebres de Clifford et Spineurs* (Univ. of Toulouse, Toulouse, 1974).

[26]R. Ablamowicz and P. Lounesto, in *Clifford Algebras and Their Applications in Mathematical Physics*, edited by J. S. R. Chisholm and L. K. Common (Reidel, Dordrecht, 1985).

[27]K. Bugajska, J. Math. Phys. **27**, 143 (1986).

[28]P. J. M. Bongaarts, Ann. Phys. **56**, 108 (1970).

[29]P. J. M. Bongaarts, in *Mathematics of Contemporary Physics*, edited by R. F. Streater (Academic, New York, 1972).

[30]P. Broadbridge and C. A. Hurst, Ann. Phys. (N.Y.) **137**, 86 (1981).

[31]H. Araki, in *Quantum Theories and Geometry*, edited by M. Cahen and M. Flato (Kluwer, Netherlands, 1988), p. 1.

[32]G. P. Wene, J. Math. Phys. **30**, 249 (1989).

[33]A. Crumeyrolle, Rep. Math. Phys. **25**, 305 (1987).

[34]S. T. Ali has suggested to us a smoothing operation to legitimize the limit.