摘 要

圖的分解問題是一種關於點集或邊集的分割的最佳化問題。本文中我們首先介紹一些圖的分解問題以及圖論中的定義和符號。

我們在第二章探討圖的最先適配分割與最先適配著色數。若圖族 $\mathcal{F}$ 中任一圖 G 均滿足 $e(G) \leq dn(G)$ 且 G 的所有導出子圖亦在 $\mathcal{F}$ 中，則我們給出 $\mathcal{F}$ 中任一圖 G 的最先適配著色數的一個上界。這個結果可應用到 d-退化圖、平面圖及外圍平面圖。

一個點加權圖 (G, c) 是一個圖 G 且 G 的每一點 v 上賦有正權重 $c(v)$ 。在第三章，我們討論點加權圖 (G, c) 的最大著色問題，此問題企圖將 $V(G)$ 分割成一些獨立集使得每一獨立集中最大點權重的總和達到最小。這是一般適當點著色問題的加權版本。對於點加權的 r-部圖，我們將給出最佳點集分割的分割數上界。而後我們將 Nordhaus-Gaddum 不等式推廣至點加權圖。我們也考慮了點加權圖的完美性質。

圖 G 的一個平衡著色是一個 $V(G)$ 的分割 $\{R, B, U\}$ 且 $|R| = |B|$ ，此處 R, B 和 U 分別代表紅色、藍色和不著色的點集。對一個有平衡著色 $\{R, B, U\}$ 的圖 G ，(R, B)-平衡分割是一個 $V(G)$ 的分割 $\mathcal{P}$ ，其滿足對任意 $S \in \mathcal{P}$ 均有導出子圖 $G[S]$ 連通和 $|S \cap R| = |S \cap B|$ 。圖 G 的平衡分解數是最小整數 ℓ ，其滿足對任意 G 的平衡著色 $\{R, B, U\}$ 均存在 (R, B)-平衡分割 $\mathcal{P}$ 使得對任何 $S \in \mathcal{P}$ 均有 $|S| \leq \ell$ 。在第四章，我們用一個較簡短的方法證明了「圖 G 的平衡分解數為 3 ，若且唯若 G 為 $\lfloor \frac{n(G)}{2} \rfloor$-連通且不為完全圖」。我們接著把平衡著色的定義從 2 色推廣到多色，並稱對應的參數為平衡 k-分解數。我們算出樹與完全多部圖的平衡 k-分解數。

圖 G 的一個奇偶邊染色是 G 的邊染色使得任意長度為正的路徑用了某個顏色奇數次。圖 G 的一個強奇偶邊染色是 G 的邊染色使得任意開放的道路用了某個顏色奇數次。圖 G 的 (強) 奇偶邊染色數是圖 G 的 (強) 奇偶邊染色所需的最少顏色數。在第5章，對於 $3 \leq m \leq n$ 且 $n \equiv 0, -1, -2 \pmod{2^{\lceil \lg m \rceil}}$ ，我們證明了 $K_{m,n}$ 的奇偶邊染色數與強奇偶邊染色數為 Hopf-Stiefel 函數，$m \circ n$ ，即對 $\ell - n < k < m$ 均滿足 $\binom{\ell}{k}$ 為偶數的最小整數 ℓ 。我們也考慮了乘積圖的奇偶與強奇偶邊染色數。

Abstract

Partition problems of graphs are optimization problems about partitions of the vertex set $V(G)$ or the edge set $E(G)$ of a graph G under some additional restrictions. We begin this thesis by introducing some partition problems, basic definitions and notation in graph theory.

We study first-fit partitions and the first-fit chromatic numbers of graphs in Chapter 2. Given a family $\mathcal{F}$ of graphs satisfying that $\mathcal{F}$ is closed under taking induced subgraphs and $e(G) \leq dn(G)$ for any graph $G \in \mathcal{F}$, where d is an arbitrary positive real number, we give an upper bound for the first-fit chromatic number of any graph in $\mathcal{F}$. This result applies to d-degenerate graphs, planar graphs, and outerplanar graphs.

A vertex-weighted graph (G, c) is a graph G with a positive weight $c(v)$ on each vertex v in G. In Chapter 3, we study the max-coloring problem of a vertex-weighted graph (G, c), which attempts to partition $V(G)$ into independent sets such that the sum of the maximum weight in each independent set is minimum. This is a weighted version of the usual vertex coloring problem of a graph. We give an upper bound for the number of sets needed in an optimal vertex partition of a vertex-weighted r-partite graph. We then derive the Nordhaus-Gaddum inequality for vertex-weighted graphs. We also consider the properties of the perfection on vertex-weighted graphs.

A balanced coloring of a graph G is a partition $\{R, B, U\}$ of $V(G)$ with $|R| = |B|$, where R, B and U stand for the sets of red, blue and uncolored vertices in G, respectively. For a graph G with a balanced coloring $\{R, B, U\}$, an (R, B)-balanced decomposition is a partition $\mathcal{P}$ of $V(G)$ such that the induced subgraph $G[S]$ is connected and $|S \cap R| = |S \cap B|$ for any S in $\mathcal{P}$. The balanced decomposition number $f(G)$ of a graph G is the minimum integer ℓ such that for any balanced coloring (R, B) of G there is an (R, B)-balanced decomposition $\mathcal{P}$ with $|S| \leq \ell$ for $S \in \mathcal{P}$. In Chapter 4, we give a shorter proof of a known result that a graph G has balanced decomposition number 3 if and only if G is $\lfloor \frac{n(G)}{2} \rfloor$-connected and

G is not a complete graph. We then extend the definition of a balanced colorin using two colors to k colors, and call the corresponding parameter the balance k-decomposition number. We compute the balanced k-decomposition numbers o trees and complete multipartite graphs.

A parity edge-coloring of a graph G is an edge-coloring of G such that any pat of positive length uses some color an odd number of times. A strong parity edge-coloring of a graph G is an edge-coloring of G such that any open walk uses som color an odd number of times. The parity (strong parity) edge-chromatic number o a graph G is the minimum number of colors used in a parity (strong parity) edge-coloring of G. In Chapter 5, we prove that, for $3 \leq m \leq n$ and $n \equiv 0, -1, -2$ (mod $2^{\lceil \lg m \rceil}$), the (strong) parity edge-chromatic number of the complete bipartite graph $K_{m,n}$ is $m \circ n$, the Hopf-Stiefel function, which is the least integer ℓ such that $\binom{\ell}{k}$ is even for each k with $\ell - n < k < m$. We also consider the parity and the strong parity edge-chromatic numbers of the products of graphs.