TRAVELING WAVE SOLUTIONS FOR A DISCRETE DIFFUSIVE EPIDEMIC MODEL

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Abstract. We study the traveling wave solutions for a discrete diffusive epidemic model. The traveling wave is a mixed of front and pulse types. We derive the existence and non-existence of traveling wave solutions of this model. The proof of existence is based on constructing a suitable pair of upper and lower solutions and the application of Schauder's fixed point theorem. By passing to the limit for a sequence of truncated problems, we are able to derive the existence of traveling waves by a delicate analysis of wave tails. Some open problems are also addressed.

1. Introduction

We consider the following lattice dynamical system

$$\dot{S}_j := \frac{dS_j}{dt} = d[S_{j+1} - 2S_j + S_{j-1}] - \beta S_j I_j, \quad j \in \mathbb{Z}, \tag{1.1a}$$

$$\dot{I}_j := \frac{dI_j}{dt} = [I_{j+1} - 2I_j + I_{j-1}] + (\beta S_j - \gamma) I_j, \quad j \in \mathbb{Z}, \tag{1.1b}$$

where d, β, γ are positive constants. Note that $(s_*, 0)$ is a constant equilibrium point of system (1.1) for any $s_* > 0$.

The system (1.1) is a spatially discrete version of the following continuous model

$$\begin{cases} u_t = du_{xx} - \beta uv, & x, t \in \mathbb{R}, \\ v_t = v_{xx} + \beta uv - \gamma v, & x, t \in \mathbb{R}, \end{cases} \tag{1.2}$$

In fact, the following kinetic system

$$\begin{cases} u_t = -\beta uv, \\ v_t = \beta uv - \gamma v, \\ w_t = \gamma v, \end{cases} \tag{1.3}$$

is the well-known classical Kermack-McKendrik model ([14]) which describes an infectious disease outbreak in a closed population consisting of susceptible

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population u, infected population v and removed population w. Here β is the transmission coefficient and γ is the recovery/remove rate.

Note that the third equation of (1.3) is de-coupled from the others. We may only consider the first two equations of (1.3). Then the model (1.2) arises when we consider the susceptible and infected populations move randomly with diffusion coefficients d and 1, respectively, in a one-dimensional environment. On the other hand, when the environment is divided into countably discrete niches, we end up with the lattice dynamical system (1.1) in which $S_j(t)$ is the susceptible population and $I_j(t)$ is the infected population at nich j at time t, respectively. We are interested in the question whether a disease can propagate spatially with a constant speed. To answer this question, one usually look for the so-called traveling wave solutions.

For a given $s^* > 0$, by a traveling wave solution of system (1.1), we mean a solution of system (1.1) in the form

$$(S_j(t), I_j(t)) = (S(\xi), I(\xi)), \quad \xi = j - ct,$$

such that $0 < S < s^*$ and $I > 0$ in $\mathbb{R}$, $S(\infty) = s^*$ and $I(\pm\infty) = 0$. Here c (the wave speed) is a constant to be determined. Also, the wave profile (S, I), as a pair of unknown positive functions, satisfies the system

$$cS' + dD[S] - \beta SI = 0 \quad \text{in } \mathbb{R}, \tag{1.4a}$$

$$cI' + D[I] + (\beta S - \gamma)I = 0 \quad \text{in } \mathbb{R}, \tag{1.4b}$$

where $D[\phi](\xi) := \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)$. Here we only require the boundary conditions

$$I(-\infty) = 0, \quad (S, I)(\infty) = (s^*, 0), \tag{1.5}$$

and leave the value of S at $-\infty$ to be free.

For the continuous model, a solution (u, v) of (1.2) is a traveling wave if $(u, v)(x, t) = (S(x - ct), I(x - ct))$ for some function (S, I) and some constant c. The traveling wave solutions of model (1.2) has been studied by Hosono and Ilyas [12]. In fact, there are tremendous works devoted to the study of traveling waves for model (1.2) and other epidemic models, including nonlocal equations with or without delays. We refer the reader to, for examples, [5, 11, 8, 10, 18, 13, 9, 17, 7, 6, 19, 16, 1, 15] and references cited therein. However, little is done for the discrete epidemic models. The purpose of this work is to study the traveling wave solutions for the discrete model (1.1).

For a given $s^* > \gamma/\beta$, we let

$$c_* := \min_{u>0} \frac{(e^u + e^{-u} - 2) + (\beta s^* - \gamma)}{u}. \tag{1.6}$$

We shall show that, under certain restrictions on s^*, there exists a traveling wave solution for the system (1.1), if $c > c_*$. However, if $c < c_*$, then system (1.4)-(1.5) does not have any positive solutions. On the other hand, if $s^* \le \gamma/\beta$, then system (1.4)-(1.5) does not have any positive solutions such that $0 < S < s^*$ in $\mathbb{R}$ for any $c > 0$.

The rest of this paper is organized as follows. In §2, we shall provide some properties of solutions to the truncated problems. By constructing a suitable pair of upper and lower solutions, we derive the existence of solutions for the truncated problems with the help of Schauder's fixed point theorem. Then we show that the existence of traveling wave solutions to (1.1) in §3. The major difficulties of deriving the existence of traveling waves are the verifications of the boundedness of I and the boundary condition of I at $\xi = -\infty$. We provide two sufficient conditions for the boundedness of I. From the boundedness of I, the boundary condition $I(-\infty) = 0$ can be readily derived. Next, §4 is devoted to the non-existence of traveling wave solutions to (1.1). Finally, some discussions are given in §5.

2. Auxiliary lemmas

Throughout this section, we let $s^* > 0$ be given such that $s^* > \gamma/\beta$ and let $c > c_*$. Under this assumption, one can easily verify that c_* defined in (1.6) is positive and for each $c > c_*$, the equation

$$p(u) := -cu + (e^u + e^{-u} - 2) + (\beta s^* - \gamma) = 0$$

has two distinct positive roots, denoted by λ and $\lambda + \delta$ for some $\delta > 0$. In addition, $p(u) < 0$ when $u \in (\lambda, \lambda + \delta)$.

2.1. **Upper- and lower-solutions.** We call $\{(\bar{S}, \bar{I}), (\underline{S}, \underline{I})\}$ a pair of upper-lower-solutions of system (1.4) on $\mathbb{R}$, if there is a finite subset $\mathcal{S}$ of $\mathbb{R}$ such that

$$c\bar{S}' + dD[\bar{S}] - \beta\bar{S}\underline{I} \le 0, \quad c\underline{S}' + dD[\underline{S}] - \beta\underline{S}\bar{I} \ge 0,$$
$$c\bar{I}' + D[\bar{I}] + (\beta\bar{S} - \gamma)\bar{I} \le 0, \quad c\underline{I}' + D[\underline{I}] + (\beta\underline{S} - \gamma)\underline{I} \ge 0$$

on $\mathbb{R} \setminus \mathcal{S}$.

We shall use an iteration process motivated by [2] to construct upper-lower-solutions. Specifically, we first construct the S-component of the upper solution $\bar{S}$, which is immediately employed to construct the I-component of the upper solution $\bar{I}$. Then $\bar{I}$ in turn is used to generate the S-component of the lower solution $\underline{S}$. Finally, we use $\underline{S}$ to construct the I-component of the lower solution $\underline{I}$.

The proof of the following lemma is trivial.

Lemma 2.1. *The function $\bar{S}(\xi) := s^*$ satisfies the inequality*

$$c\bar{S}' + dD[\bar{S}] - \beta\bar{S}I_0 \le 0$$

for all $\xi \in \mathbb{R}$ for any nonnegative function I_0.

Next, we have

Lemma 2.2. *The function* $\bar{I}(\xi) := e^{-\lambda\xi}$ *satisfies the equation*

$$c\bar{I}' + D[\bar{I}] + (\beta\bar{S} - \gamma)\bar{I} = 0, \tag{2.1}$$

for all $\xi \in \mathbb{R}$.

Proof. Since $p(\lambda) = 0$, it follows that

$$c\bar{I}' + D[\bar{I}] + (\beta\bar{S} - \gamma)\bar{I} = p(\lambda)\bar{I} = 0, \forall \xi \in \mathbb{R}.$$

Hence the lemma follows. □

Select $\alpha \in (0, \lambda)$ small enough such that $c\alpha - d\,(e^{\alpha} + e^{-\alpha} - 2) > 0$. Since $e^{(\alpha-\lambda)\xi} \to 0$ as $\xi \to \infty$, there exists a $\xi_1 > 0$ such that

$$e^{(\alpha-\lambda)\xi} \le \beta^{-1}\left[c\alpha - d\left(e^{\alpha} + e^{-\alpha} - 2\right)\right],\ \forall\ \xi \ge \xi_1,$$

so that

$$e^{-\alpha\xi}\left[c\alpha - d\left(e^{\alpha} + e^{-\alpha} - 2\right)\right] \ge \beta\bar{I}(\xi),\ \forall\ \xi \ge \xi_1. \tag{2.2}$$

Set $M := s^* e^{\alpha\xi_1}$. Then $M > s^*$ and we have

Lemma 2.3. *The function* $\underline{S}(\xi) := \max\{0, s^* - Me^{-\alpha\xi}\}$ *satisfies the inequality*

$$c\underline{S}' + dD[\underline{S}] - \beta\underline{S}\bar{I} \ge 0 \tag{2.3}$$

for all $\xi \ne \xi_1$.

Proof. For $\xi < \xi_1$, since $\underline{S} \equiv 0$ in $(-\infty, \xi_1)$, it is obvious that the inequality (2.3) holds. For $\xi > \xi_1$, $\underline{S}(\xi) = s^* - Me^{-\alpha\xi}$. Using (2.2) and the fact that $M > s^*$, we deduce that

$$c\underline{S}' + dD[\underline{S}] \ge \left[c\alpha - d\left(e^{\alpha} + e^{-\alpha} - 2\right)\right] \cdot Me^{-\alpha\xi} \ge \beta s^* \bar{I} \ge \beta\underline{S}\bar{I}.$$

Hence (2.3) holds. □

Choose $0 < \eta < \min\{\alpha, \delta\}$. Then $\eta - \alpha < 0$ and $p(\lambda + \eta) < 0$. Select

$$L > \max\{M/s^*, -\beta M/p(\lambda + \eta)\}. \tag{2.4}$$

Set $\xi_2 := 1/\eta \cdot \ln L$. Then $\xi_2 > \xi_1 > 0$, since $\xi_1 = 1/\alpha \cdot \ln(M/s^*)$, $L > M/s^* > 1$, and $\alpha > \eta > 0$.

Lemma 2.4. *The function* $\underline{I}(\xi) := \max\{0, e^{-\lambda\xi} - Le^{-(\lambda+\eta)\xi}\}$ *satisfies the inequality*

$$c\underline{I}' + D[\underline{I}] + (\beta\underline{S} - \gamma)\underline{I} \ge 0, \tag{2.5}$$

for all $\xi \ne \xi_2$.

Proof. For $\xi < \xi_2$, the inequality (2.4) holds immediately since $\underline{I} \equiv 0$ in $(-\infty, \xi_2)$. For $\xi > \xi_2$, $\underline{I} = \bar{I} - Le^{-(\lambda+\eta)\xi}$ and $\underline{S} = s^* - Me^{-\alpha\xi}$. A simple computation gives that

$$\begin{aligned}
\underline{I}' &= \bar{I}' + (\lambda+\eta)Le^{-(\lambda+\eta)\xi},\\
D[\underline{I}] &\geq D[\bar{I}] - LD[e^{-(\lambda+\eta)\xi}]\\
&= D[\bar{I}] - \left[e^{(\lambda+\eta)} + e^{-(\lambda+\eta)} - 2\right] Le^{-(\lambda+\eta)\xi},\\
(\beta\underline{S}-\gamma)\underline{I} &= \left(\beta s^* - \gamma - \beta M e^{-\alpha\xi}\right)\left(\bar{I} - Le^{-(\lambda+\eta)\xi}\right)\\
&\geq (\beta s^* - \gamma)\bar{I} - (\beta s^* - \gamma)Le^{-(\lambda+\eta)\xi} - \beta M e^{-(\lambda+\alpha)\xi}.
\end{aligned}$$

Together with (2.1) and the definition of p, we get

$$c\underline{I}' + D[\underline{I}] + (\beta\underline{S}-\gamma)\underline{I} \geq e^{-(\lambda+\eta)\xi}[-p(\lambda+\eta)L - \beta M e^{(\eta-\alpha)\xi}] \geq 0,$$

where we have used $\eta - \alpha < 0$ and $L > -\beta M/p(\lambda+\eta)$. The proof of this lemma is therefore completed. □

We conclude that $\{(\bar{S},\bar{I}),(\underline{S},\underline{I})\}$, constructed in Lemmas 2.1-2.4, is a pair of upper-lower-solutions of system (1.4) on $\mathbb{R}$.

2.2. **A truncated problem.** In this subsection, we let $l > \xi_2$ be fixed and consider the following truncated problem

$$cS' + dD[S] - \beta SI = 0 \quad \text{in } [-l,l], \tag{2.6a}$$

$$cI' + D[I] + (\beta S - \gamma)I = 0 \quad \text{in } [-l,l], \tag{2.6b}$$

together with the boundary conditions

$$(S,I) = (0,0) \text{ on } (-\infty,-l), \tag{2.7a}$$

$$(S,I) = (\underline{S},\underline{I}) \text{ on } [l,\infty). \tag{2.7b}$$

Here

$$\begin{aligned}
S'(-l) &:= \lim_{h\searrow 0}(S(-l+h)-S(-l))/h,\ S'(l) := \lim_{h\searrow 0}(S(l)-S(l-h))/h,\\
I'(-l) &:= \lim_{h\searrow 0}(I(-l+h)-I(-l))/h,\ I'(l) := \lim_{h\searrow 0}(I(l)-I(l-h))/h.
\end{aligned}$$

For convenience, we let

$$\begin{aligned}
X &:= C([-l,l]) \times C([-l,l]),\\
Y &:= [C^1([-l,l]) \times C^1([-l,l])] \cap [C([-l,\infty)) \times C([-l,\infty))].
\end{aligned}$$

We shall apply Schauder's fixed point theorem to show that there exists a pair of functions $(S,I) \in Y$ satisfying (2.6)-(2.7). For this, we set

$$E := \{(S,I) \in X \mid \underline{S} \leq S \leq \bar{S} \text{ and } \underline{I} \leq I \leq \bar{I} \text{ in } [-l,l]\},$$

which is a closed convex set in the Banach space X equipped with the norm $\|(f_1,f_2)\|_X = \|f_1\|_{C([-l,l])} + \|f_2\|_{C([-l,l])}$. Besides, since $\underline{S}$ and $\underline{I}$ are non-negative, it follows that S and I are non-negative on $[-l,l]$ for any $(S,I) \in E$.

Lemma 2.5. *For a given* $(S_0, I_0) \in E$*, there exists a unique solution* $(S, I) \in E$ *to the problem*

$$cS' + dD[S] - \beta S I_0 = 0 \quad \text{in } [-l, l], \tag{2.8a}$$

$$cI' + D[I] + \beta S_0 I_0 - \gamma I = 0 \quad \text{in } [-l, l], \tag{2.8b}$$

$$(S, I) = (0, 0) \text{ on } (-\infty, -l), \tag{2.8c}$$

$$(S, I) = (\underline{S}, \underline{I}) \text{ on } (l, \infty), \tag{2.8d}$$

such that $(S, I) \in Y$.

Proof. First, we employ the monotone iteration technique to show the existence result.

For a given fixed $(S_0, I_0) \in E$, we choose a positive constant μ such that

$$\mu \geq \max \left\{ \frac{2d}{c} + \frac{\beta}{c} \cdot \max_{\xi \in [-l,l]} I_0(\xi), \frac{2}{c} + \frac{\gamma}{c} \right\}$$

and introduce the functionals

$$H_1(S, I_0)(\xi) := \mu S(\xi) + \frac{d}{c} D[S](\xi) - \frac{\beta}{c} S(\xi) I_0(\xi),$$

$$H_2(I, S_0, I_0)(\xi) := \mu I(\xi) + \frac{1}{c} D[I](\xi) + \frac{\beta}{c} S_0(\xi) I_0(\xi) - \frac{\gamma}{c} I(\xi).$$

Using H_1 and H_2, we can rewrite (2.8a) and (2.8b) as follows:

$$-S' + \mu S = H_1(S, I_0), \quad -I' + \mu I = H_2(I, S_0, I_0).$$

It follows that a pair of functions $(S, I) \in Y$ satisfying (2.8c)-(2.8d) is a solution of (2.8) if and only if (S, I) satisfies the following integral system

$$S(\xi) = T_1^l[S, I_0](\xi) \quad \text{for } \xi \in [-l, l], \tag{2.9a}$$

$$I(\xi) = T_2^l[I, S_0, I_0](\xi) \quad \text{for } \xi \in [-l, l], \tag{2.9b}$$

where

$$T_1^l[S, I_0](\xi) := e^{\mu(\xi - l)} \underline{S}(l) + \int_\xi^l e^{\mu(\xi - z)} H_1(S, I_0)(z) dz,$$

$$T_2^l[I, S_0, I_0](\xi) := e^{\mu(\xi - l)} \underline{I}(l) + \int_\xi^l e^{\mu(\xi - z)} H_2(I, S_0, I_0)(z) dz.$$

One can easily check that the following monotonic properties hold:

$S \geq \hat{S}$ in $[-l-1, l+1] \Rightarrow H_1(S, I_0) \geq H_1(\hat{S}, I_0)$ in $[-l, l]$,

$I \geq \hat{I}$ in $[-l-1, l+1] \Rightarrow H_2(I, S_0, I_0) \geq H_2(\hat{I}, S_0, I_0)$ in $[-l, l]$

$I_0 \geq \hat{I}_0$ in $[-l, l] \Rightarrow H_1(S, \hat{I}_0) \geq H_1(S, I_0)$ in $[-l, l]$,

$S_0 \geq \hat{S}_0, I_0 \geq \hat{I}_0$ in $[-l, l] \Rightarrow H_2(I, S_0, I_0) \geq H_2(I, \hat{S}_0, \hat{I}_0)$ in $[-l, l]$.

Moreover, we have

$$\bar{S}(\xi) \geq T_1^l[\bar{S}, I_0](\xi), \quad \underline{S}(\xi) \leq T_1^l[\underline{S}, I_0](\xi), \tag{2.10}$$

$$\bar{I}(\xi) \geq T_2^l[\bar{I}, S_0, I_0](\xi), \quad \underline{I}(\xi) \leq T_2^l[\underline{I}, S_0, I_0](\xi), \tag{2.11}$$

for all $\xi \in [-l, \infty)$. Here we have defined the functions (S_0, I_0), $T_1^l[S, I_0]$ and $T_2^l[I, S_0, I_0]$ outside $[-l, l]$ by (2.7).

Now we define inductively that

$$S_1 = \underline{S},\ I_1 = \underline{I},\ S_{k+1} = T_1^l[S_k, I_0],\ I_{k+1} = T_2^l[I_k, S_0, I_0],\ k \in \mathbb{N}.$$

Using the inequalities (2.10) and (2.11) and the monotonic property of H_i, $i = 1, 2$, one can easily show that

$$\underline{S} \le S_k \le S_{k+1} \le \bar{S}, \quad \underline{I} \le I_k \le I_{k+1} \le \bar{I}$$

on $[-l, l]$ for each $k \in \mathbb{N}$. It follows that the limit

$$(S(\xi), I(\xi)) := (\lim_{k\to\infty} S_k(\xi), \lim_{k\to\infty} I_k(\xi)), \quad \forall \xi \in \mathbb{R},$$

exists, $(S, I) \in E$, and (S, I) satisfies (2.8c)-(2.8d). Further, applying Lebesgue's dominated convergence theorem, we see that (S, I) satisfies (2.9) and hence is a solution of (2.8). In addition, one can easily show that $(S, I) \in Y$.

The proof of uniqueness is standard and we omit it. Hence we have completed the proof of this lemma. □

Now we define the mapping $T : E \to E$ by

$$T(S_0, I_0) = (S, I), \quad (S_0, I_0) \in E,$$

where (S, I) is the unique solution of the boundary value problem (2.8). It is clear that any fixed point of T is a solution of the problem (2.6)-(2.7). By a standard argument, one can show that T is a continuous precompact mapping of E into E. Since its proof is standard, we omit it here. Hence Schauder's fixed point theorem asserts that T has a fixed point, which is a solution of system (2.6)-(2.7). So we have proved the following theorem.

Theorem 1. For each $l > \xi_2$, system (2.6)-(2.7) admits a unique solution $(S, I) \in Y$ such that

$$0 \le \underline{S} \le S \le s^*, \quad 0 \le \underline{I} \le I \le \bar{I} \tag{2.12}$$

over $[-l, \infty)$.

3. Existence of traveling wave solutions

In this section, we shall establish the existence of traveling wave solutions. First, we prove

Theorem 2. Let $s^* > 0$ be given such that $s^* > \gamma/\beta$. If $c > c_*$, then the system (1.4) admits a positive solution (S, I) such that $0 < S < s^*$ in $\mathbb{R}$, $I > 0$ in $\mathbb{R}$, and

$$(S, I)(\infty) = (s^*, 0). \tag{3.1}$$

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $(0,\infty)$ such that $l_n \to \infty$ as $n \to \infty$, and let (S_n, I_n), $n \in \mathbb{N}$, be the solution of system (2.6)-(2.7) established in Theorem 1 for $l = l_n$. For any fixed $N \in \mathbb{N}$, since the function $\bar{I}$ is bounded above in $[-l_N, l_N]$, it follows from (2.12) that the sequences

$$\{S_n\}_{n\geq N}, \{I_n\}_{n\geq N}, \text{ and } \{S_nI_n\}_{n\geq N}$$

are uniformly bounded in $[-l_N, l_N]$. Then, by (2.6), we infer that the sequences

$$\{S_n'\}_{n\geq N} \text{ and } \{I_n'\}_{n\geq N}$$

are also uniformly bounded in $[-l_N, l_N]$. Using (2.6), we can express S_n'' and I_n'' in terms of S_n, I_n, S_n' and I_n'. Consequently, the sequences

$$\{S_n''\}_{n\geq N} \text{ and } \{I_n''\}_{n\geq N}$$

are uniformly bounded in $[-l_N + 1, l_N - 1]$. With the aid of Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence $\{(S_{n_j}, I_{n_j})\}$ of $\{(S_n, I_n)\}$ such that

$$S_{n_j} \to S,\ S_{n_j}' \to S', \quad I_{n_j} \to I,\ I_{n_j}' \to I',$$

uniformly in any compact interval in $\mathbb{R}$ as $j \to \infty$, for some functions S and I in $C^1(\mathbb{R})$. Then it is easy to see that (S, I) is a nonnegative solution of system (1.4) and satisfies (2.12) over $\mathbb{R}$. From definitions of $\underline{S}$ and $\bar{I}$, we see that $\underline{S}(\xi) \to s^*$ and $\bar{I}(\xi) \to 0$ as $\xi \to \infty$. This, together with (2.12), implies (3.1).

Now, we claim that $0 < S < s^*$ and $I > 0$ over $\mathbb{R}$. For contradiction, we assume that $I(\tilde{\xi_1}) = 0$ for some $\tilde{\xi_1} \in \mathbb{R}$. Then $I'(\tilde{\xi_1}) = 0$. This, together with (1.4b) and the non-negativity of I, gives that $I(\tilde{\xi_1}+1) = I(\tilde{\xi_1}-1) = 0$. Hence, by induction, we obtain that $I(\tilde{\xi_1} + j) = 0$ for all $j \in \mathbb{Z}$, which contradicts the fact that $I \geq \underline{I} > 0$ on (ξ_2, ∞). Hence $I > 0$ over $\mathbb{R}$. By a similar argument, we also have $S > 0$ over $\mathbb{R}$. To prove $S < s^*$ over $\mathbb{R}$, we also use a contradictory argument and assume that $S(\tilde{\xi_2}) = s^*$ for some $\tilde{\xi_2} \in \mathbb{R}$. In this case, $S'(\tilde{\xi_2}) = 0$, $D[S](\tilde{\xi_2}) \leq 0$, and $S(\tilde{\xi_2})I(\tilde{\xi_2}) > 0$. This contradicts (1.4a) at $\xi = \tilde{\xi_2}$. Hence $S < s^*$ over $\mathbb{R}$ and the theorem is proved. □

We note that any solution (S, I) obtained in Theorem 2 satisfies the property

$$\int_{-\infty}^{\infty} S(\tau)I(\tau)d\tau < \infty. \tag{3.2}$$

Indeed, integrating equation (1.4a) from y to $\xi > y$ gives that

$$\begin{aligned}\beta\int_y^\xi S(\tau)I(\tau)d\tau \;&=\; d\Big[\int_\xi^{\xi+1} S(\tau)d\tau - \int_{\xi-1}^{\xi} S(\tau)d\tau \\ &\quad + \int_{y-1}^{y} S(\tau)d\tau - \int_y^{y+1} S(\tau)d\tau\Big] + c[S(\xi) - S(y)].\end{aligned}$$

Hence (3.2) follows by using the boundedness of S over $\mathbb{R}$. In particular, we also have

$$\liminf_{\xi\to-\infty} S(\xi)I(\xi) = 0.$$

To derive the existence of traveling wave solutions, we need to show that $I(-\infty) = 0$.

To this end, we first provide two conditions to ensure the boundedness of I as follows.

Lemma 3.1. *Suppose that $s^* > \gamma/\beta$. Let (S, I) be a solution of* (1.4) *obtained in Theorem 2 for a given $c > c_*$. If $c > 1$, then I is bounded.*

Proof. Following [19], we consider the solution of truncated problem. Let (S, I) be the solution of system (2.6)-(2.7) for a given $l > 0$. Suppose that I takes the maximum value at ξ_0 for some $\xi_0 > -l$. First, we integrate (2.6b) from $-l$ to $\xi > -l$ to get

$$\begin{aligned}
&\beta\int_{-l}^{\xi} S(\tau)I(\tau)d\tau\\
=\;& cS(\xi) - cS(-l) - d\int_{-l}^{-l+1} S(\tau)d\tau + d\int_{\xi}^{\xi+1} S(\tau)d\tau - d\int_{\xi-1}^{\xi} S(\tau)d\tau.
\end{aligned}$$

Letting $\xi \to \infty$, we obtain

$$\beta\int_{-l}^{\infty} S(\tau)I(\tau)d\tau = cs^* - cS(-l) - d\int_{-l}^{-l+1} S(\tau)d\tau \le cs^*.$$

Similarly, by integrating (2.6b) over $[\xi_0, \infty)$, we obtain

$$\begin{aligned}
&cI(\xi_0)\\
=\;& \beta\int_{\xi_0}^{\infty} S(\tau)I(\tau)d\tau - \gamma\int_{\xi_0}^{\infty} I(\tau)d\tau + \int_{\xi_0-1}^{\xi_0} I(\tau)d\tau - \int_{\xi_0}^{\xi_0+1} I(\tau)d\tau\\
\le\;& cs^* + I(\xi_0).
\end{aligned}$$

It follows that $I(\xi_0)$ is uniformly bounded (independent of l) when $c > 1$. Hence I is bounded in $\mathbb{R}$ for any solution (S, I) of (1.4) obtained in Theorem 2, if $c > 1$. □

Remark 3.2. In fact, under the assumption that

$$\beta s^* - \gamma \ge 2 + \ln[(\sqrt{5}+1)/2] - \sqrt{5}, \tag{3.3}$$

we have $c_* \ge 1$. Indeed, setting

$$g(u) := e^u + e^{-u} - 2 + \beta s^* - \gamma - u,$$

we easily verify that g is strictly convex such that $g(0^+) = \beta s^* - \gamma > 0$, $g(\infty) = \infty$ and g has a unique minimal point at u_0 with $e^{u_0} = (\sqrt{5}+1)/2$. Then the condition (3.3) ensures that $g(u_0) \ge 0$. Hence $c_* \ge 1$.

Lemma 3.3. *Suppose that $s^* > \gamma/\beta$. Let (S, I) be a solution of (1.4) obtained in Theorem 2 for a given $c > c_*$. If $\liminf_{x\to-\infty} S(x) > 0$, then I is bounded.*

Proof. By integrating equation (1.4b) from y to ξ, we have

$$\gamma \int_y^\xi I(\tau)d\tau = \beta \int_y^\xi S(\tau)I(\tau)d\tau + \Big[\int_\xi^{\xi+1} I(\tau)d\tau - \int_{\xi-1}^{\xi} I(\tau)d\tau + \int_{y-1}^{y} I(\tau)d\tau - \int_y^{y+1} I(\tau)d\tau\Big] + c[I(\xi) - I(y)] \tag{3.4}$$

for any $y < \xi$. Due to $\liminf_{x\to-\infty} S(x) > 0$, we have $S \geq \hat{s}$ in $\mathbb{R}$ for some $\hat{s} > 0$. Then

$$\hat{s}\int_y^\infty I(\tau)d\tau \leq \int_y^\infty S(\tau)I(\tau)d\tau \leq \int_{-\infty}^\infty S(\tau)I(\tau)d\tau$$

for all $y \in \mathbb{R}$. It follows from (3.2) that I is integrable over $\mathbb{R}$.

Finally, sending $\xi \to \infty$ in (3.4) gives that

$$\begin{aligned} & cI(y) \\ = \;& \beta \int_y^\infty S(\tau)I(\tau)d\tau + \Big[\int_{y-1}^{y} I(\tau)d\tau - \int_y^{y+1} I(\tau)d\tau\Big] - \gamma\int_y^\infty I(\tau)d\tau \\ \leq \;& \beta \int_{-\infty}^\infty S(\tau)I(\tau)d\tau + \int_{-\infty}^\infty I(\tau)d\tau \end{aligned}$$

for all $y \in \mathbb{R}$. Hence I is bounded in $\mathbb{R}$ and the proof is complete. □

In fact, the boundary condition $I(-\infty) = 0$ is assured by the boundedness of I as follows.

Lemma 3.4. *Suppose that I is bounded in $\mathbb{R}$. Then $I(-\infty) = 0$.*

Proof. Assume for contradiction that $\sigma := \limsup_{\xi\to-\infty} I(\xi) > 0$.

First, integrating equation (1.4b) from y to ∞ (or recall from the proof of Lemma 3.3) gives that

$$\begin{aligned} & \gamma\int_y^\infty I(\tau)d\tau \\ = \;& \beta \int_y^\infty S(\tau)I(\tau)d\tau + \Big[\int_{y-1}^{y} I(\tau)d\tau - \int_y^{y+1} I(\tau)d\tau\Big] - cI(y), \end{aligned}$$

which, together with the boundedness of I over $\mathbb{R}$ and (3.2), yields that the improper integral

$$\int_{-\infty}^\infty I(\tau)d\tau$$

is convergent.

Next, due to the boundedness of S and I over $\mathbb{R}$, we see from (1.4b) that I' is also bounded in $\mathbb{R}$ and so there exists a positive constant K such that $|I'(\xi)| \leq K$ for all $\xi \in \mathbb{R}$. Choose a sequence $\{\xi_n\} \searrow -\infty$ such that $\xi_n - \xi_{n+1} > \sigma/(4K)$ and $I(\xi_n) \geq \sigma/2$. For $\xi \in (\xi_n - \sigma/(4K), \xi_n)$, we have

$$I(\xi_n) - I(\xi) = \int_{\xi}^{\xi_n} I'(\tau)d\tau \leq K(\xi_n - \xi),$$

and so

$$I(\xi) \geq I(\xi_n) - K(\xi_n - \xi) \geq \frac{\sigma}{4}.$$

Hence

$$\int_{-\infty}^{\infty} I(\tau)d\tau \geq \sum_{n=1}^{\infty} \int_{\xi_n - \sigma/(4K)}^{\xi_n} I(\tau)d\tau = \infty,$$

a contradiction. This proves the lemma. □

Lemma 3.5. *Suppose that the limit $s_* := S(-\infty)$ exists. Then $s_* < s^*$.*

Proof. Integrating (1.4a) from $-\infty$ to $x \in \mathbb{R}$, we obtain

$$c[S(x) - s_*] + d\left[\int_x^{x+1} S(\tau)d\tau - \int_{x-1}^{x} S(\tau)d\tau\right] = \beta\int_{-\infty}^{x} S(\tau)I(\tau)d\tau > 0.$$

Letting $x \to \infty$ and recalling that $S(\infty) = s^*$, we deduce that $s_* < s^*$. Hence the lemma follows. □

Combining the above lemmas, in particular we have the following theorem on the existence of traveling waves.

Theorem 3. Suppose that $s^* > \gamma/\beta$ such that (3.3) holds. Then (1.1) has a traveling wave solution for any wave speed $c > c_*$.

4. Non-existence of traveling wave solutions

In this section, we shall deal with the non-existence of positive solution (S, I) of (1.4) such that

$$0 < S < s^*,\ I > 0,\ S(+\infty) = s^*,\ I(\pm\infty) = 0. \tag{4.1}$$

First, we consider the case when $s^* > \gamma/\beta$. Suppose that there exists a positive solution (S, I) of (1.4) satisfying (4.1) for some $c \in (0, c_*)$. Set $z(x) := I'(x)/I(x)$. Then it is easy to see from (1.4b) that z satisfies the equation

$$cz(x) + \left(e^{\int_x^{x+1} z(\xi)d\xi} + e^{\int_x^{x-1} z(\xi)d\xi} - 2\right) + \beta S(x) - \gamma = 0.$$

Since the limit

$$\lim_{x\to\infty}[\beta S(x) - \gamma] = \beta s^* - \gamma > 0,$$

it follows from a fundamental theory of [4] that the limit $\nu := \lim_{x\to\infty} z(x)$ exists and satisfies the equation

$$c\nu + (e^{\nu} + e^{-\nu} - 2) + \beta s^* - \gamma = 0.$$

This is impossible, due to $c \in (0, c_*)$ and the definition of c_*.

Next, we consider the case when $s^* \le \gamma/\beta$. Suppose that there exists a positive solution (S, I) of (1.4) satisfying (4.1) for some $c > 0$. In this case, we first recall from (3.2) and (3.4) that

$$\int_{-\infty}^{\infty} S(x)I(x)dx < \infty, \quad \int_{-\infty}^{\infty} I(x)dx < \infty.$$

Integrating (1.4b) over $\mathbb{R}$, we obtain that

$$0 = \int_{-\infty}^{\infty} [\beta S(x) - \gamma]I(x)dx \le \beta \int_{-\infty}^{\infty} [S(x) - s^*]I(x)dx \le 0,$$

using $S < s^* \le \gamma/\beta$. This implies that

$$\int_{-\infty}^{\infty} [S(x) - s^*]I(x)dx = 0.$$

It then follows from $I > 0$ in $\mathbb{R}$ that $S \equiv s^*$ in $\mathbb{R}$, a contradiction.

We summarize the above discussions as the following theorem.

Theorem 4. There is no traveling wave solution of (1.1) if either (i) $s^* > \gamma/\beta$ and $c \in (0, c_*)$, or, (ii) $s^* \le \gamma/\beta$ and $c > 0$.

5. Discussions

In this paper, we study the existence and non-existence of traveling wave solutions for a discrete diffusive epidemic model. Our traveling wave is a mixed of front (S component) and pulse (I component) types. However, the definition of traveling wave here is weaker than the standard one. Usually, for a traveling wave, we require $(S, I)(-\infty) = (s_*, 0)$ for some $s_* \in (0, s^*)$.

In the continuous model (1.2), it is easy to see that S is monotone. Indeed, for a positive traveling wave (c, S, I) of (1.2) with wave speed $c > 0$, the wave profile (S, I) satisfies

$$dS'' + cS' - \beta SI = 0 \quad \text{in } \mathbb{R}, \tag{5.1}$$

$$I'' + cI' + \beta SI - \gamma I = 0 \quad \text{in } \mathbb{R}. \tag{5.2}$$

Then it is easy to see from (5.1) that any critical point of S is a strictly minimal point. Hence there is at most one critical point of S. Suppose that $S'(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Then $S'(\xi) < 0$ for all $\xi < \xi_0$. This implies that

$$dS''(\xi) = \beta S(\xi)I(\xi) - cS'(\xi) > 0, \ \forall \, \xi < \xi_0,$$

and so $S(\xi) \to \infty$ as $\xi \to -\infty$, a contradiction. Hence there is no critical point of S and so S must be monotone. Therefore, the limit $S(-\infty)$ always exists for the model (1.2).

However, for the discrete model (1.1), we are not sure whether S is monotone. It might be oscillatory.

On the other hand, one may expect that I has only one peak, namely, $I' > 0$ on $(-\infty, \xi_0)$ and $I' < 0$ on (ξ_0, ∞) for some $\xi_0 \in \mathbb{R}$. Actually, due to

(5.2), $S(\xi) \geq \gamma/\beta$ for any peak ξ of I. Therefore, if $S(\infty) > S(-\infty) \geq \gamma/\beta$, then I has only one peak for the model (1.2).

For our discrete model, we have $I' < 0$ in a neighborhood of $x = \infty$. For this, we consider the quantity $z(x) := I'(x)/I(x)$. Then it is easy to see from (1.4b) that z satisfies the equation

$$cz(x) + \left(e^{\int_x^{x+1} z(\xi)d\xi} + e^{\int_x^{x-1} z(\xi)d\xi} - 2\right) + \beta S(x) - \gamma = 0.$$

Since the limit

$$\lim_{x\to\infty} [\beta S(x) - \gamma] = \beta s^* - \gamma > 0,$$

it follows from a fundamental theory of [4, 3] that the limit $\nu := \lim_{x\to\infty} z(x)$ exists and satisfies the equation

$$c\nu + (e^{\nu} + e^{-\nu} - 2) + \beta s^* - \gamma = 0.$$

Since $c > 0$, we see that $\nu < 0$. This implies that $I'(x) < 0$ in a neighborhood of $x = \infty$. Similarly, if the limit $s_* = S(-\infty)$ exists and $\beta s_* - \gamma < 0$, then we also have $I'(x) > 0$ in a neighborhood of $x = -\infty$.

However, we are not sure whether there is only one peak for our discrete model (1.1).

Finally, combining Theorems 3 and 4, we see that c_* is the minimal speed of traveling waves of (1.1) when $s^* > \gamma/\beta$ such that (3.3) holds. However, the existence of traveling wave solution for $c = c_*$ is still open.

We leave these questions for the future studies.

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